STOUFFER

Invariants of Linear Differential Equations with Applications to Ruled Surfaces in Five-Dimensional Space

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INVARIANTS OF LINEAR DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO RULED SURFACES IN FIVE-DIMENSIONAL SPACE

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

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Contents.

In Wilczynski's *Projective differential geometry of curves and ruled surfaces*, the general theory of non-developable ruled surfaces in ordinary space is put into relation with the theory of a system of two linear homogeneous differential equations of the second order. In this paper the same notion is extended to five-dimensional space. It is shown that the projective differential properties of every ruled surface in five dimensions, which does not belong to a certain exceptional class, can be obtained by studying a system of two linear homogeneous differential equations of the third order.

The author takes pleasure in acknowledging his indebtedness to Professor Wilczynski for helpful advice and general guidance in the preparation of this paper.

1. *Transformations of the general system.*

Let us consider the system of linear homogeneous differential equations

$$(1) \qquad y_i^{(m)} + \sum_{\ell=0}^{m-1} \sum_{k=1}^{n} \binom{m}{\ell} p_{ik\ell}\, y_k^{(\ell)} = 0, \qquad (i = 1,2,\ldots n),$$

where

$$y_k = \frac{d^{\ell} y_k}{dx^{\ell}},$$

and where $p_{ik\ell}$ are functions of the independent variable x. Wilczynski * has shown that the most general point transformation which converts this system into another of the same form is given by the equations

$$(2) \qquad y_k = \sum_{\lambda=1}^{n} \alpha_{k\lambda}(\xi)\eta_\lambda, \qquad x = f(\xi), \qquad (k = 1,2,\ldots n),$$

* Wilczynski, *Projective differential geometry of curves and ruled surfaces*, Chap. I.

where $\alpha_{k\lambda}$ and f are arbitrary functions of ξ, and where the determinant

$$|\alpha_{k\lambda}|, \quad (k,\lambda = 1,2,\ldots n),$$

does not vanish identically.

A function of the coefficients of (1) and of their derivatives which has the same value for (1) as for any system derived from (1) by transformations of the form (2) is called an absolute *invariant*. If the function contains also the dependent variables and their derivatives, it is called a *covariant*. Functions which remain invariant under all transformations of the form (2) for which the independent variable remains unaltered, so that only the dependent variables are transformed, are called *seminvariants* and *semi-covariants*.

The transformations (2) form an infinite continuous group which, according to Lie's terminology, is defined by differential equations. Lie* has shown that the invariants of such a group may always be determined as the solutions of a complete system of partial differential equations which is obtained by equating to zero the symbol of the most general infinitesimal transformation of the group.

These same considerations apply to the transformations

$$y_k = \sum_{\lambda=1}^{n} \alpha_{k\lambda}(\xi)\eta_\lambda \tag{3}$$

of the dependent variables alone. They form a subgroup of the infinite group defined by (2).

The transformation (3) converts equations (1) into a new system whose coefficients $\pi_{\lambda\mu\nu}$ are expressed in terms of the old coefficients by means of the equations**

* Lie, Mathematische Annalen, vol. 24.

**Wilczynski, loc. cit., p. 93. The difference between our formula (4) and that of Wilczynski arises from the change of notation involved in introducing binomial coefficients in (1).

$$(4)\qquad \Delta\pi_{\lambda\mu\nu} = \sum_{i=1}^{n} A_{i\lambda}\,[\alpha_{i}^{(m-\nu)} + \sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{m-\nu}{\tau} p_{i,k,\nu+\tau}\,\alpha_{k\mu}^{(\tau)}],$$
$$(\lambda,\mu = 1,2,\ldots n;\ \nu = 0,1,\ldots m-1),$$

where $A_{i\lambda}$ is the minor of $\alpha_{i\lambda}$ in the determinant

$$\Delta = |\alpha_{i\lambda}|, \qquad (i,\lambda = 1,2,\ldots n).$$

The most general infinitesimal transformation of the form (3) will be obtained by putting

$$\alpha_{ii} = 1+\varphi_{ii}(x)\delta t,\quad \alpha_{ik} = \varphi_{ik}(x)\delta t,\quad (i \neq k,\ i,k = 1,2,..n),$$

where δt is an infinitesimal and φ_{ik} is an arbitrary function of x. If we substitute these values of α_{ik} in (4) and denote the infinitesimal difference $\pi_{\lambda\mu\nu}-p_{\lambda\mu\nu}$ by $\delta p_{\lambda\mu\nu}$, we find for the infinitesimal transformations of $p_{\lambda\mu\nu}$

$$(5)\quad \frac{\delta p_{\lambda\mu\nu}}{\delta t} = \sum_{k=1}^{n}(\varphi_{k\mu}p_{\lambda k\nu}-\varphi_{\lambda k}p_{k\mu\nu}) + \sum_{k=1}^{n}\sum_{\tau=1}^{m-1-\nu}\binom{m-\nu}{\tau}\varphi_{k\mu}^{(\tau)}p_{\lambda k,\nu+\tau} + \varphi_{\lambda\mu}^{(m-\nu)},$$
$$(\lambda,\mu = 1,2,..n;\quad \nu = 0,1,..m-1).$$

The infinitesimal transformations of the derivatives of $p_{\lambda\mu\nu}$ can be obtained directly from these equations by differentiation.

2. *Calculation of the seminvariants for $m = 3$; $n=2$.*

Let us now confine ourselves to the special case of a system of two linear homogeneous differential equations of the third order. We shall assume the equations to be written in the form

$$(A)\quad \begin{aligned} &y''' + 3p_{11}y'' + 3p_{12}z'' + 3q_{11}y' + 3q_{12}z' + r_{11}y + r_{12}z = 0,\\ &z''' + 3p_{21}y'' + 3p_{22}z'' + 3q_{21}y' + 3q_{22}z' + r_{21}y + r_{22}z = 0.\end{aligned}$$

By comparing these equations with the general equations (1), we see that we have put

$$p_{\lambda\mu 2} = p_{\lambda\mu},\quad p_{\lambda\mu 1} = q_{\lambda\mu},\quad p_{\lambda\mu 0} = r_{\lambda\mu}.$$

The general equations (5) give for the infinitesimal transformations of the

coefficients of (A) the results

$$
\begin{aligned}
&\frac{\delta p_{\lambda\mu}}{\delta t} = \sum_{k=1}^{2} (\varphi_{k\mu} p_{\lambda k} - \varphi_{\lambda k} p_{k\mu}) + \varphi'_{\lambda\mu}, \\
(6)\qquad &\frac{\delta q_{\lambda\mu}}{\delta t} = \sum_{k=1}^{2} (\varphi_{k\mu} q_{\lambda k} - \varphi_{\lambda k} q_{k\mu} + 2\varphi'_{k\mu} p_{\lambda k}) + \varphi''_{\lambda\mu}, \\
&\frac{\delta r_{\lambda\mu}}{\delta t} = \sum_{k=1}^{2} (\varphi_{k\mu} r_{\lambda k} - \varphi_{\lambda k} r_{k\mu} + 3\varphi'_{k\mu} q_{\lambda k} + 3\varphi''_{k\mu} p_{\lambda k}) + \varphi'''_{\lambda\mu}.
\end{aligned}
$$

We shall proceed immediately to the calculation of the seminvariants of the system (A). Let f be any seminvariant depending only on the arguments $p_{\lambda\mu}$, $p'_{\lambda\mu}$, $q_{\lambda\mu}$. The infinitesimal transformations of the dependent variables give to f the increment

$$
\delta f = \sum_{\lambda\mu} \left(\frac{\partial f}{\partial p_{\lambda\mu}} \delta p_{\lambda\mu} + \frac{\partial f}{\partial p'_{\lambda\mu}} \delta p'_{\lambda\mu} + \frac{\partial f}{\partial q_{\lambda\mu}} \delta q_{\lambda\mu} \right).
$$

This expression must vanish for all values of the arbitrary functions φ_{rs}, φ'_{rs}, and φ''_{rs}. Consequently the coefficients of these functions, when equated to zero, give a system of partial differential equations of which f must be a solution. Moreover, Lie's theory of infinite groups tells us that this is a complete system, and that any solution of it is a seminvariant.

Writing out this system we find

$$
\begin{aligned}
&(a)\quad \frac{\partial f}{\partial p'_{\nu s}} + \frac{\partial f}{\partial q_{\nu s}} = 0, \\
(7)\quad &(b)\quad \frac{\partial f}{\partial p_{\nu s}} + \sum_{\lambda=1}^{2} \left(p_{\lambda\nu} \frac{\partial f}{\partial p'_{\lambda s}} - p_{s\lambda} \frac{\partial f}{\partial p'_{\nu\lambda}} + 2 p_{\lambda\nu} \frac{\partial f}{\partial q_{\lambda s}} \right) = 0, \\
&(c)\quad \sum_{\lambda=1}^{2} \left(p_{\lambda\nu} \frac{\partial f}{\partial p_{\lambda s}} - p_{s\lambda} \frac{\partial f}{\partial p_{\nu\lambda}} + p'_{\lambda\nu} \frac{\partial f}{\partial p'_{\lambda s}} - p'_{s\lambda} \frac{\partial f}{\partial p'_{\nu\lambda}} + q_{\lambda\nu} \frac{\partial f}{\partial q_{\lambda s}} - q_{s\lambda} \frac{\partial f}{\partial q_{\nu\lambda}} \right) = 0, \\
&\qquad (\nu, s = 1, 2).
\end{aligned}
$$

It contains twelve variables and twelve equations, but only ten of the equations are independent. One relation between them is evident. For, the left member of the equation of (7) (c) obtained by putting r = s = 1 is, except for its sign, the same as that obtained by putting r = s = 2.

This relation is especially important as it persists in the equations for the seminvariants containing higher derivatives of $p_{\lambda\mu}$, $q_{\lambda\mu}$, $r_{\lambda\mu}$. The second relation between equations(7) does not so persist.

The general theory of partial differential equations tells us that the system (7) has two independent solutions. We shall proceed to find them. Equations (a) evidently have the four solutions

$$p'_{rs} - q_{rs}.$$

Equations (b) show that $p'_{rs} - q_{rs}$ and p_{rs} can occur only in the four combinations

$$u_{ik} = p'_{ik} - q_{ik} + \sum_{j=1}^{2} p_{ij}\, p_{jk}, \qquad (i, k = 1,2). \tag{8}$$

Finally, equations (c) are found to have the two independent solutions

$$I = u_{11} + u_{22}, \qquad J = u_{11}u_{22} - u_{12}u_{21}. \tag{9}$$

These are the two seminvariants involving only the arguments p_{rs}, p'_{rs}, q_{rs}.

Let us next find the seminvariants involving also p''_{rs}, q'_{rs}, r_{rs}. The system of differential equations for these seminvariants contains 24 variables and 16 equations. Only 15 of the equations are independent as there is a relation here similar to the first one in the preceeding case. The second relation does not persist. Therefore, there are just 9 independent seminvariants in this case.

By the method employed above we find that twelve of the equations are satisfied by the independent quantities

$$u_{ik}, \quad v_{ik} = u'_{ik} + \sum_{j=1}^{2} (p_{ij}u_{jk} - p_{jk}u_{ij}),$$

$$\begin{aligned} t_{ik} = p''_{ik} - 3q'_{ik} + 2r_{ik} - \sum_{j=1}^{2} (p_{ij}p'_{jk} - 3p_{jk}p'_{ij} + 2p_{jk}q_{ij} \\ - 3p_{ij}u_{jk} - p_{jk}u_{ij}), \end{aligned} \tag{10}$$

$$(i, k = 1, 2).$$

The remaining three independent equations are satisfied by the following nine combinations of the quantities (10)

$$
(11)\qquad
\begin{aligned}
I &= u_{11} + u_{22}, & J &= u_{11}u_{22}-u_{12}u_{21},\\
I' &= v_{11} + v_{22}, & G &= v_{11}v_{22}-v_{12}v_{21},\\
H &= t_{11} + t_{22}, & K &= t_{11}t_{22}-t_{12}t_{21},\\
J' &= u_{11}v_{22}+u_{22}v_{11}-u_{12}v_{21}-u_{21}v_{12}, & &\\
L &= u_{11}t_{22}+u_{22}t_{11}-u_{12}t_{21}-u_{21}t_{12}, & &\\
M &= v_{11}t_{22}+v_{22}t_{11}-v_{12}t_{21}-v_{21}t_{12}. & &
\end{aligned}
$$

That these nine seminvariants are independent is easily seen if we put $p_{ik} = 0$ and notice that the functional determinant

$$
\frac{\partial(I, I', J, J', G, H, K, L, M)}{\partial(q_{11}, q_{12}, q_{21}, q_{22}, q'_{11}, q'_{12}, q'_{21}, r_{11}, r_{12})} = 32 r_{21}(q'_{21}r_{12}-q'_{12}r_{21}) \begin{vmatrix} q_{11}-q_{22}, & q_{12}, & q_{21} \\ q'_{11}-q'_{22}, & q_{12}, & q_{21} \\ r_{11}-r_{22}, & r_{12}, & r_{21} \end{vmatrix}
$$

does not vanish identically.

In order to determine the seminvariants involving the further variables p'''_{rs}, q''_{rs}, r'_{rs}, we must solve a system of differential equations containing four more equations and twelve more variables than the system in the preceeding case. The one relation between the equations maintains itself, so that we have just eight new seminvariants. Seven of these are I'', J'', G', H', K', L', M'. We shall find the eighth without actually integrating the equations.

The quantities u_{ik}, v_{ik}, t_{ik} are cogredient under a transformation of the dependent variables. This fact is made evident by the infinitesimal transformations of these quantities

$$
(12)\qquad
\begin{aligned}
\frac{\delta u_{ik}}{\delta t} &= \sum_{j=1}^{2}(u_{ij}\,\varphi_{jk} - u_{jk}\,\varphi_{ij}),\\
\frac{\delta v_{ik}}{\delta t} &= \sum_{j=1}^{2}(v_{ij}\,\varphi_{jk} - v_{jk}\,\varphi_{ij}),\\
\frac{\delta t_{ik}}{\delta t} &= \sum_{j=1}^{2}(t_{ij}\,\varphi_{jk} - t_{jk}\,\varphi_{ij}).
\end{aligned}
$$

Now we have found in (11) that certain combinations of the u_{ik}'s and the p_{ik}'s are seminvariants. If we replace u_{ik} by v_{ik} or by t_{ik} in these combinations, we must still have seminvariants. In particular, let us form the quantities w_{ik} from the v_{ik}'s in the same way that the latter are formed from the u_{ik}'s, so that we have

$$(13) \qquad w_{ik} = v'_{ik} + \sum_{j=1}^{2}(p_{ij}v_{jk}-p_{jk}v_{ij}).$$

Then the combination

$$(14) \qquad N = w_{11}w_{22}-w_{12}w_{21}$$

is obtained by replacing u_{ik} by v_{ik} in G and is, therefore, a seminvariant. That it is independent of the seminvariants (11) and their first derivatives can be proved in the same way that the seminvariants (11) were proved independent.

We have now found 17 independent seminvariants involving only $p_{ik}, p'_{ik}, p''_{ik}, p'''_{ik}, q_{ik}, q'_{ik}, q''_{ik}, r_{ik}, r'_{ik}$. Any other seminvariant depending on these quantities alone must be expressible in terms of the seminvariants which we have found.

The system of equations for the seminvariants involving also the next higher derivatives of p_{ik}, q_{ik}, r_{ik} contains four more independent equations and twelve more variables than the system in the last case. Therefore, there are eight more seminvariants. These are evidently the next higher derivatives of I, J, G, H, K, L, M, N. If we proceed in this way, we find that *all the seminvariants of the system (A) are functions of I, J, G, H, K, L, M, N and of their derivatives.*

Let us write the transformation (3) of the dependent variables in the form

$$y = \alpha\bar{y}+\beta\bar{z}, \quad z = \gamma\bar{y}+\delta\bar{z}, \quad \alpha\delta-\beta\gamma = \Delta.$$

If we make this transformation and denote the coefficients of the new system by $\bar{p}_{ik}, \bar{q}_{ik}, \bar{r}_{ik}$, we find

$$
(15)\qquad
\begin{aligned}
\Delta\bar{p}_{11} &= \alpha'\delta-\gamma'\beta + p_{11}\alpha\delta+p_{12}\gamma\delta-p_{21}\alpha\beta-p_{22}\gamma\beta,\\
\Delta\bar{p}_{12} &= \beta'\delta-\delta'\beta + p_{11}\beta\delta+p_{12}\delta^2-p_{21}\beta^2-p_{22}\delta\beta,
\end{aligned}
$$

$$
(16)\qquad
\begin{aligned}
\Delta\bar{q}_{11} &= \alpha''\delta-\gamma''\beta +2p_{11}\alpha'\delta+2p_{12}\gamma'\delta-2p_{21}\alpha'\beta-2p_{22}\gamma'\beta\\
&\qquad + q_{11}\alpha\delta+q_{12}\gamma\delta-q_{21}\alpha\beta-q_{22}\gamma\beta,\\
\Delta\bar{q}_{12} &= \beta''\delta-\delta''\beta +2p_{11}\beta'\delta+2p_{12}\delta'\delta-2p_{21}\beta'\beta-2p_{22}\delta'\beta\\
&\qquad + q_{11}\beta\delta+q_{12}\delta^2-q_{21}\beta^2-q_{22}\delta\beta,
\end{aligned}
$$

$$
(17)\qquad
\begin{aligned}
\Delta\bar{r}_{11} &= \alpha'''\delta-\gamma'''\beta +3p_{11}\alpha''\delta+3p_{12}\gamma''\delta-3p_{21}\alpha''\beta-3p_{22}\gamma''\beta\\
&\qquad +3q_{11}\alpha'\delta+3q_{12}\gamma'\delta-3q_{21}\alpha'\beta-3q_{22}\gamma'\beta\\
&\qquad + r_{11}\alpha\delta+r_{12}\gamma\delta-r_{21}\alpha\beta-r_{22}\gamma\beta,\\
\Delta\bar{r}_{12} &= \beta'''\delta-\delta'''\beta+3p_{11}\beta''\delta+3p_{12}\delta''\delta-3p_{21}\beta''\beta-3p_{22}\delta''\beta\\
&\qquad +3q_{11}\beta'\delta+3q_{12}\delta'\delta-3q_{21}\beta'\beta-3q_{22}\delta'\beta\\
&\qquad + r_{11}\beta\delta+r_{12}\delta^2-r_{21}\beta^2-r_{22}\delta\beta.
\end{aligned}
$$

The values of $\bar{p}_{21}$ and $\bar{p}_{22}$ can be obtained from the values of $\bar{p}_{12}$ and $\bar{p}_{11}$ respectively by interchanging simultaneously the quantities α and δ, β and γ, and the subscripts 1 and 2. By the same changes we can derive $\bar{q}_{21}$ and $\bar{q}_{22}$ and $\bar{r}_{12}$ and $\bar{r}_{22}$ from (16) and (17).

From these equations and equations (8) we find

$$
\begin{aligned}
\Delta\bar{u}_{11} &= u_{11}\alpha\delta+u_{12}\gamma\delta-u_{21}\alpha\beta-u_{22}\gamma\beta,\\
\Delta\bar{u}_{12} &= u_{11}\beta\delta+u_{12}\delta^2-u_{21}\beta^2-u_{22}\delta\beta,\\
\Delta\bar{u}_{21} &= -u_{11}\alpha\gamma-u_{12}\gamma^2+u_{21}\alpha^2+u_{22}\alpha\gamma,\\
\Delta\bar{u}_{22} &= -u_{11}\beta\gamma-u_{12}\gamma\delta+u_{21}\alpha\beta+u_{22}\alpha\delta.
\end{aligned}
$$

Since v_{ik}, t_{ik}, w_{ik} are cogredient with u_{ik}, the equations for $\bar{v}_{ik}, \bar{t}_{ik}, \bar{w}_{ik}$ are of the same form as (18).

3. Calculation of the invariants.

Evidently the invariants of system (A) are such functions of the seminvariants as remain unchanged after an arbitrary transformation of the independent variable. Therefore, in order to calculate the invariants it is necessary to find the effect of such a transformation on the seminvariants.

If we make the transformation

$$\xi = \xi(x)$$

and denote the coefficients of the new system by $\bar{p}_{ik}, \bar{q}_{ik}, \bar{r}_{ik}$, we find

$$
\begin{aligned}
&\bar{p}_{11} = \frac{1}{\xi'}(p_{11} + \eta), && \bar{p}_{12} = \frac{1}{\xi'} p_{12}, \\
&\bar{p}_{21} = \frac{1}{\xi'} p_{21}, && \bar{p}_{22} = \frac{1}{\xi'}(p_{22} + \eta), \\
(19)\qquad &\bar{q}_{11} = \frac{1}{(\xi')^2}\left(q_{11} + p_{11}\eta + \frac{\eta' + \eta^2}{3}\right), && \bar{q}_{12} = \frac{1}{(\xi')^2}(q_{12} + p_{12}\eta), \\
&\bar{q}_{21} = \frac{1}{(\xi')^2}(q_{21} + p_{21}\eta), && \bar{q}_{22} = \frac{1}{(\xi')^2}\left(q_{22} + p_{22}\eta + \frac{\eta' + \eta^2}{3}\right), \\
&\bar{r}_{ik} = \frac{1}{(\xi')^3} r_{ik},
\end{aligned}
$$

where

$$\eta = \frac{\xi''}{\xi'}.$$

In these equations the expressions for $\bar{p}_{11}$ and $\bar{p}_{12}$ are of the same form as the expressions for $\bar{p}_{22}$ and $\bar{p}_{21}$ respectively. The same is true for $\bar{q}_{ik}$ and $\bar{r}_{ik}$ and is, therefore, true also for $\bar{u}_{ik}$, $\bar{v}_{ik}$, $\bar{t}_{ik}$, $\bar{w}_{ik}$. Consequently, we need to indicate only the effect of the transformation on those quantities whose subscripts are 11 and 12.

From (8) and the formula obtained by differentiating $\bar{p}_{ik}$ with respect to ξ we obtain

(20) $$\bar{u}_{11} = \frac{1}{(\xi')^2}\left(u_{11} + \tfrac{2}{3}\mu\right), \qquad \bar{u}_{12} = \frac{1}{(\xi')^2} u_{12},$$

where

$$\mu = \eta' - \tfrac{1}{2}\eta^2.$$

From (20) we find

(21) $$\bar{I} = \frac{1}{(\xi')^2}\left(I + \tfrac{4}{3}\mu\right), \quad \bar{J} = \frac{1}{(\xi')^4}\left(J + \tfrac{2}{3}\mu I + \tfrac{4}{9}\mu^2\right).$$

These equations give at once the relative invariant of weight 4,

(22) $$\Theta_4 = I^2 - 4J.^{*}$$

From equations (10) we find

(23) $$\bar{v}_{11} = \frac{1}{(\xi')^3}\left(v_{11} - 2u_{11}\eta + \tfrac{2}{3}\mu' - \tfrac{4}{3}\mu\eta\right), \quad \bar{v}_{12} = \frac{1}{(\xi')^3}\left(v_{12} - 2u_{12}\eta\right),$$

and

(24) $$\bar{t}_{11} = \frac{1}{(\xi')^3} t_{11}, \quad \bar{t}_{12} = \frac{1}{(\xi')^3} t_{12}.$$

The last equations show at once that H and K are relative invariants of weights 3 and 6 respectively. We shall denote them by Θ_3 and Θ_6. Finally, from (13) we find

(25) $$\begin{aligned} \bar{w}_{11} &= \frac{1}{(\xi')^4}\left(w_{11} + 5u_{11}\eta^2 - 2u_{11}\mu - 5v_{11}\eta + \tfrac{2}{3}\mu'' - \tfrac{10}{3}\mu'\eta - \tfrac{4}{3}\mu^2 + \tfrac{10}{3}\mu\eta^2\right), \\ \bar{w}_{12} &= \frac{1}{(\xi')^4}\left(w_{12} + 5u_{12}\eta^2 - 2u_{12}\mu - 5v_{12}\eta\right). \end{aligned}$$

In computing the invariants we shall need the infinitesimal transformations of the quantities whose finite transformations we have found. The most general infinitesimal transformation of the independent variable is found by putting

(26) $$\xi(x) = x + \phi(x)\,\delta t, \quad \delta(x) = \phi(x)\,\delta t,$$

where $\phi(x)$ is an arbitrary function and δt an infinitesimal. For this value of $\xi(x)$ we find

$$\xi' = 1 + \phi'(x)\,\delta t, \quad \eta = \phi''(x)\,\delta t, \quad \mu = \phi'''(x)\,\delta t, \quad \mu' = \phi^{(4)}\delta t, \cdots.$$

Substituting these values in (19) we have

(27) $$\begin{gathered} \delta p_{11} = (\phi'' - \phi' p_{11})\,\delta t, \qquad \delta p_{12} = -\phi' p_{12}\,\delta t, \\ \delta q_{11} = (\tfrac{1}{3}\phi''' + \phi'' p_{11} - 2\phi' q_{11})\,\delta t, \quad \delta q_{12} = (\phi'' p_{12} - 2\phi' q_{12})\,\delta t, \\ \delta N_{ik} = -3\phi' N_{ik}\,\delta t. \end{gathered}$$

From (20) we find

(28) $$\delta u_{11} = (\tfrac{2}{3}\phi''' - 2\phi' u_{11})\,\delta t, \quad \delta u_{12} = -2\phi' u_{12}\,\delta t,$$

whence

(29) $$\delta I = (\tfrac{4}{3}\phi''' - 2\phi' I)\,\delta t, \quad \delta J = (\tfrac{2}{3}\phi''' I - 4\phi' J)\,\delta t.$$

*For the notion of weight, see Wilczynski, loc. cit., p.106.

Again, from (23),

(30) $$\delta v_{11} = (\tfrac{2}{3}\varphi^{(4)} - 2\varphi'' u_{11} - 3\varphi' v_{11})\,\delta t, \quad \delta v_{12} = (-2\varphi'' u_{12} - 3\varphi' v_{12})\,\delta t,$$

whence

(31) $$\delta \mathcal{G} = (\tfrac{2}{3}\varphi^{(4)} I' - 2\varphi'' J' - 6\varphi' \mathcal{G})\,\delta t.$$

Further

(32) $$\delta t_{ik} = -3\varphi' t_{ik}\,\delta t,$$

so that we have

(33) $$\delta L = (\tfrac{2}{3}\varphi''' H - 5\varphi' L)\,\delta t, \qquad \delta M = (\tfrac{2}{3}\varphi^{(4)} H - 2\varphi'' L - 6\varphi' M)\,\delta t.$$

Finally, from (25),

(34) $$\delta w_{11} = (\tfrac{2}{3}\varphi^{(5)} - 2\varphi''' u_{11} - 5\varphi'' v_{11} - 4\varphi' w_{11})\,\delta t, \qquad \delta w_{12} = (-2\varphi''' u_{12} - 5\varphi'' v_{12} - 4\varphi' w_{12})\,\delta t.$$

whence

(35) $$\delta N = (\tfrac{2}{3}\varphi^{(5)} I'' - 2\varphi''' J'' + 4\varphi''' \mathcal{G} - 5\varphi'' \mathcal{G}' - 8\varphi' N)\,\delta t.$$

By applying the formula

$$\delta\left(\frac{df}{dx}\right) = \frac{d}{dx}(\delta f) - \varphi' \frac{df}{dx}\,\delta t,^{*}$$

we can find the infinitesimal transformations of the derivatives of the seminvariants. In particular, we have

(36) $$\delta I' = (\tfrac{4}{3}\varphi^{(4)} - 2\varphi'' I - 3\varphi' I')\,\delta t, \qquad \delta J' = (\tfrac{2}{3}\varphi^{(4)} I + \tfrac{2}{3}\varphi''' I' - 4\varphi'' J - 5\varphi' J')\,\delta t.$$

We have found three invariants, Θ_3, Θ_4, and Θ_6, of weights 3, 4, and 6 respectively. We can easily find one of weight 5. We have

$$\frac{\delta(IH)}{\delta t} = \tfrac{4}{3}\varphi''' H - 5\varphi' IH.$$

From this expression and (33) we see at once that

(37) $$\Theta_5 = IH - 2L$$

is such an invariant.

In a similar manner we can obtain an invariant of weight 10. We have

$$\frac{\delta(4\mathcal{G} - I'^2)}{\delta t} = 4\varphi''(II' - 2J') - 6\varphi'(4\mathcal{G} - I'^2),$$

* Wilczynski, loc. cit., p.106.

$$\frac{\delta(II'-2J')}{\delta t} = -2\Phi''(I^2-4J) - 5\Phi'(II'-2J').$$

By the elimination of Φ'' from these equations we obtain the invariant

(38) $$\Theta_{10} = (I^2-4J)(4G-I'^2)+(II'-2J')^2.$$

The invariants which we have found thus far involve only the quantities $p_{ik}, p'_{ik}, p''_{ik}, q_{ik}, q'_{ik}, r_{ik}$. Let us now find all the independent invariants depending on these quantities alone. Such invariants must be functions of the seminvariants

(39) $$I, I', J, J', G, H, K, L, M.$$

We can find the system of partial differential equations for the invariants depending on the quantities (39) in the same way that we have found the systems for the seminvariants. This system will be found to contain 4 independent equations and has, therefore, five solutions. Hence there are 6 such relative invariants. We have found five and by integrating the equations we easily find another of weight 10,

(40) $$\vartheta_{10} = (I'H-2M)(I^2-4J)-(IH-2L)(II'-2J').$$

Thus, we have found the following six relative invariants

(41) $$\Theta_3,\ \Theta_4,\ \Theta_5,\ \Theta_6,\ \Theta_{10},\ \vartheta_{10}.$$

That they are independent can be easily verified by considering them in the above order and noticing that each contains a seminvariant not in a preceeding one.

We can prove by direct substitution that

(42) $$\Theta_8 = \begin{vmatrix} u_{11}-u_{22}, & u_{12}, & u_{21} \\ t_{11}-t_{22}, & t_{12}, & t_{21} \\ v_{11}-v_{22}, & v_{12}, & v_{21} \end{vmatrix}$$

is also an invariant. It depends on the same quantities as the invariants (41) and therefore must be expressible in terms of them. We shall find this expression later and we shall see that Θ_8 is not a rational function of the invariants (41).

We can easily find a number of invariants involving the next higher derivatives of p_{ik}, q_{ik}, r_{ik}. The Jacobian of any two invariants Θ_μ and Θ_ν is an invariant of weight $\mu+\nu+1$. By the Jacobian of Θ_μ and Θ_ν we mean the factor $\mu\Theta_\mu\Theta_\nu' - \nu\Theta_\nu\Theta_\mu'$ which occurs in the numerator of the derivative of the absolute invariant $\dfrac{\Theta_\nu^\mu}{\Theta_\mu^\nu}$. We have the following Jacobians

$$\begin{aligned}
\vartheta_8 &= 4\Theta_4\Theta_3' - 3\Theta_3\Theta_4', \\
\Phi_{10} &= 4\Theta_4\Theta_5' - 5\Theta_5\Theta_4', \\
\Theta_{11} &= 2\Theta_4\Theta_6' - 3\Theta_6\Theta_4', \qquad (43) \\
\Theta_{15} &= 2\Theta_4\Theta_{10}' - 5\Theta_{10}\Theta_4', \\
\vartheta_{15} &= 2\Theta_4\vartheta_{10}' - 5\vartheta_{10}\Theta_4'.
\end{aligned}$$

From any invariant Θ_m we can always deduce another of weight 2m+2,

$$\Theta_{m.1} = 2m\Theta_m''\Theta_m - (2m+1)(\Theta_m')^2 + \tfrac{3}{2}m^2 I\,\Theta_m^2,$$

which is called the quadriderivative* of Θ_m. In particular we have

(44) $$\Theta_{4.1} = 8\Theta_4''\Theta_4 - 9(\Theta_4')^2 + 24 I\,\Theta_4^2.$$

We have further the invariant

(45) $$\Theta_9 = \begin{vmatrix} u_{11}-u_{22}, & u_{12}, & u_{21} \\ v_{11}-v_{22}, & v_{12}, & v_{21} \\ w_{11}-w_{22}, & w_{12}, & w_{21} \end{vmatrix}$$

similar to Θ_8. If the invariants (41),(43),(44),(45) are considered in the proper order, it is easily seen that they are independent.

The system of differential equations for the invariants depending on the seminvariants (39) and their first derivatives shows that there are just 13 independent relative invariants of this kind. We have found all of them.

*Wilczynski, loc. cit., p. 112.

4. The complete system of invariants.

Before proving that we have found a complete system of invariants, it will be necessary to show how the system (A) can be transformed so that it will have certain special properties. Let us make the transformation

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta, \quad \alpha\delta - \beta\gamma \neq 0.$$

The system (A) goes into another of the same form whose terms in η'' and ζ'' will vanish if (α, γ) and (β, δ) are chosen as two pairs of solutions of the system of equations

$$\begin{aligned} \rho' &= -(p_{11}\rho + p_{12}\sigma), \\ \sigma' &= -(p_{21}\rho + p_{22}\sigma). \end{aligned} \tag{46}$$

Since $\alpha\delta - \beta\gamma \neq 0$, (α, γ) and (β, δ) must be two independent systems of solutions. As two such systems always exist, the system (A) can always be converted into another for which $p_{ik} = 0$. We shall call this the <u>semi-canonical form of the system (A)</u>.

Let us now assume that the system is given in its semi-canonical form. The general transformation

$$\xi = \xi(x), \quad y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta,$$

converts it into a system which is also in the semi-canonical form provided that

$$\alpha\xi'' + \alpha'\xi' = \beta\xi'' + \beta'\xi' = \gamma\xi'' + \gamma'\xi' = \delta\xi'' + \delta'\xi' = 0.$$

Thus $\alpha, \beta, \gamma, \delta$ must have the values

$$\alpha = \frac{a}{\xi'}, \quad \beta = \frac{b}{\xi'}, \quad \gamma = \frac{c}{\xi'}, \quad \delta = \frac{d}{\xi},$$

where a, b, c, d are constants whose determinant ad - bc does not vanish. Therefore, <u>the most general transformation which leaves the semi-canonical form invariant is given by the equations</u>

$$\xi = \xi(x), \quad \eta = (ay + bz)\xi', \quad \zeta = (cy + dz)\xi', \tag{47}$$

<u>where $\xi(x)$ is an arbitrary function and a, b, c, d are arbitrary constants.</u>

We can reduce the system (A) to another special form which will be found very useful. Let us choose the coefficients $\alpha, \beta, \gamma, \delta$ of the general transformation of the dependent variables in such a way that $\frac{\beta}{\delta}$ and $\frac{\alpha}{\gamma}$ are the two roots of the equation

(48) $$-u_{21}\lambda^2 + (u_{11} - u_{22})\lambda + u_{12} = 0.$$

Then equations (18) show that

$$\bar{u}_{12} = \bar{u}_{21} = 0.$$

Since $\alpha\delta - \beta\gamma$ must not vanish, the roots of (48) must be distinct. This means that the transformation is not possible if

$$(u_{11} - u_{22})^2 + 4u_{12}u_{21} = \Theta_4 = 0.$$

In exactly the same way we can reduce the system (A) to another for which

$$\bar{t}_{12} = \bar{t}_{21} = 0,$$

if

$$(t_{11} - t_{22})^2 + 4t_{12}t_{21} = \Theta_3^2 - 4\Theta_6 \neq 0$$

Let us assume that system (A) has been reduced to a form for which either $u_{12} = u_{21} = 0$ or $t_{12} = t_{21} = 0$. Equations (18), (20), and (24) show that these conditions will be left invariant by a transformation of the form

$$y = \alpha\eta, \quad z = \delta\zeta, \quad \xi = \xi(x),$$

where α, δ and ξ are arbitrary functions of x. After such a transformation has been made, the new system will have its coefficients $\bar{p}_{11}$ and $\bar{p}_{22}$ equal to zero if α and δ satisfy the equations

$$\alpha' + p_{11}\alpha + \frac{\xi''}{\xi'}\alpha = 0,$$
$$\delta' + p_{22}\delta + \frac{\xi''}{\xi'}\delta = 0.$$

By solving these equations we see that a system for which either $u_{12} = u_{21} = 0$ or $t_{12} = t_{21} = 0$ is converted into another having also $p_{11} = p_{22} = 0$ by the transformation

$$y = \frac{a}{\xi'} e^{-\int p_{11} dx} \eta , \quad z = \frac{b}{\xi'} e^{-\int p_{22} dx} \zeta . \qquad \xi = \xi(x)$$

where ξ is an arbitrary function and a and b are arbitrary constants.

We can still select the arbitrary function $\xi(x)$ so as to make any of the non-vanishing invariants equal to unity. In particular, the equation

$$\bar{\Theta}_4 = \frac{1}{(\xi')^4} \Theta_4$$

shows that we shall have

$$\Theta_4 = (u_{11} - u_{22})^2 = 1 ,$$

if we put

$$\xi = \int \sqrt{u_{11} - u_{22}}\, dx + \text{const.}$$

We can now easily find the expression for Θ_8 in terms of the invariants (41). Let us assume that the system (A) has been transformed into one for which

$$u_{12} = u_{21} = 0, \qquad p_{11} = p_{22} = 0, \qquad u_{11} - u_{22} = 1.$$

A relation between the invariants will not be disturbed. Under these conditions we have

$$\begin{aligned}
\Theta_8 &= t_{12} p_{21} + t_{21} p_{12}, \\
\Theta_5 &= t_{11} - t_{22}, \\
\Theta_3^2 - 4\Theta_6 &= 4 t_{12} t_{21} + (t_{11} - t_{22})^2, \\
\Theta_{10} &= 4 p_{12} p_{21}, \\
\vartheta_{10} &= 2(t_{12} p_{21} - t_{21} p_{12}),
\end{aligned}$$

whence

$$4\Theta_8^2 \Theta_4 = \Theta_{10} (\Theta_3^2 \Theta_4 - 4\Theta_6 \Theta_4 - \Theta_5^2) + \vartheta_{10}^2 .$$

In deducing this equation we have introduced $\Theta_4 = 1$ in such a way as to make the whole expression isobaric.

Let us now assume that the eight independent invariants

(49) $\quad \Theta_3, \Theta_4, \Theta_5, \Theta_8, \Theta_9, \Theta_{4\cdot 1}, \Theta_{10}, \vartheta_{10},$

are given as functions of x. We shall show that a system of equations of the form (A) having these invariants is determined uniquely except for transformations of the form

$$y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta.$$

For the present we shall assume that Θ_4 and Θ_{10} are both different from zero.

Suppose that

$$(50)\qquad \begin{aligned} &y''' + 3p_{11}\, y'' + 3p_{12}\, z'' + 3q_{11}\, y' + 3q_{12}\, z' + r_{11}\, y + r_{12}\, z = 0, \\ &z''' + 3p_{21}\, y'' + 3p_{22}\, z'' + 3q_{21}\, y' + 3q_{22}\, z' + r_{21}\, y + r_{22}\, z = 0, \end{aligned}$$

is a system whose coefficients are unknown but whose invariants (49) are the arbitrarily chosen functions of x. Since we are taking Θ_4 different from zero, we can transform (50) into a system satisfying the conditions

$$(51)\qquad u_{12} = u_{21} = p_{11} = p_{22} = 0.$$

Then the invariants (49) become

$$(52)\qquad \begin{aligned} &\Theta_4 = (u_{11} - u_{22})^2, \\ &\Theta_{4.1} = 8\Theta_4\Theta_4'' - 9(\Theta_4')^2 + 24(u_{11} + u_{22})\Theta_4^2, \\ &\Theta_{10} = 4p_{12}p_{21}(u_{11} - u_{22})^4, \\ &\Theta_9 = (u_{11} - u_{22})^3(p_{12}'p_{21} - p_{21}'p_{12}), \\ &\Theta_3 = 3(u_{11}' + u_{22}') - 6(p_{12}'p_{21} + p_{21}'p_{12}) + 2(v_{11} + v_{22}), \\ &\Theta_5 = (u_{11} - u_{22})\{3(u_{11}' - u_{22}') + 2(p_{12}'p_{21} - p_{21}'p_{12}) + 2(v_{11} - v_{22})\}, \\ &\vartheta_{10} = 2(u_{11} - u_{22})^3\{(2p_{21}'' - 3p_{21}u_{11} - p_{21}u_{22} + 2p_{21}q_{22} - 2v_{21})p_{12} \\ &\qquad - (2p_{12}'' - p_{12}u_{11} - 3p_{12}u_{22} + 2p_{12}q_{11} - 2v_{12})p_{21}\} \\ &\Theta_8 = -(u_{11} - u_{22})^2\{(2p_{21}'' - 3p_{21}u_{11} - p_{21}u_{22} + 2p_{21}q_{22} - 2v_{21})p_{12} \\ &\qquad + (2p_{12}'' - p_{12}u_{11} - 3p_{12}u_{22} + 2p_{12}q_{11} - 2v_{12})p_{21}\} \end{aligned}$$

These expressions together with the conditions (51) enable us to express the coefficients of (50) in terms of the invariants (49). Thus we find

$$
\begin{aligned}
&p_{11} = 0\,, \qquad\qquad p_{22} = 0\,,\\
&p_{12} = \epsilon' C \frac{\sqrt{\Theta_{10}}}{2\Theta_4} e^{2\epsilon\int \sqrt{\Theta_4}\frac{\Theta_9}{\Theta_{10}}dx},\\
&p_{21} = \epsilon' \frac{1}{C} \frac{\sqrt{\Theta_{10}}}{2\Theta_4} e^{-2\epsilon\int \sqrt{\Theta_4}\frac{\Theta_9}{\Theta_{10}}dx},\\
&q_{12} = p'_{12}\,, \qquad\qquad q_{21} = p'_{21}\,,\\
(53)\qquad &48q_{11} = \frac{1}{\Theta_4^2}\left\{12\Theta_{10} - \Theta_{4.1} + 8\Theta_4\Theta_4'' - 9(\Theta_4')^2\right\} - 24\epsilon\sqrt{\Theta_4},\\
&48q_{22} = \frac{1}{\Theta_4^2}\left\{12\Theta_{10} - \Theta_{4.1} + 8\Theta_4\Theta_4'' - 9(\Theta_4')^2\right\} + 24\epsilon\sqrt{\Theta_4},\\
&4r_{11} = \Theta_3 + \frac{\Theta_5}{\epsilon\sqrt{\Theta_4}} + 8p_{12}p'_{21} + 4p'_{12}p_{21} - 6u_{11},\\
&4r_{22} = \Theta_3 - \frac{\Theta_5}{\epsilon\sqrt{\Theta_4}} + 8p'_{12}p_{21} + 4p_{12}p'_{21} - 6u_{22},\\
&4r_{12} = \frac{\Theta_8}{\Theta_4 p_{21}} + \frac{\vartheta_{10}}{2\epsilon\Theta_4^{3/2}p_{21}} + 4p''_{12} + 4p_{12}q_{11} - 2u_{11}p_{12} - 6u_{22}p_{12},\\
&4r_{21} = \frac{\Theta_8}{\Theta_4 p_{12}} - \frac{\vartheta_{10}}{2\epsilon\Theta_4^{3/2}p_{12}} + 4p''_{21} + 4p_{21}q_{22} - 6u_{11}p_{21} - 2u_{22}p_{21}
\end{aligned}
$$

where C is an arbitrary constant and $\epsilon = \pm 1$ and $\epsilon' = \pm 1$. The method of expressing q_{12}, q_{21}, and r_{ik} in terms of the invariants directly is obvious.

We can easily remove the arbitrary constant and the ambiguous signs. The transformation

$$y = k\eta\,, \qquad z = l\zeta\,,$$

where

$$\frac{k}{l} = \epsilon' C,$$

will convert a system whose coefficients are given by (53) with C equal to any constant and ϵ' either +1 or -1, into the system whose coefficients are also given by (53) with C = 1, ϵ' = 1. Again the transformation

$$y = \zeta\,, \qquad z = \eta\,,$$

will convert the system whose coefficients are given by (53) with $\epsilon = -1$, into the system whose coefficients are given by the same equations with $\epsilon = 1$.

We have now proved the following fundamental theorem. If the invariants (49) are given as arbitrary functions of x, but so that Θ_4 and Θ_{10} are different from zero, they determine a system of differential equations of the form (A) uniquely except for transformations of the dependent variables of the form

$$\bar{y} = \alpha y + \beta z, \quad \bar{z} = \gamma y + \delta z,$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions of x.

We shall see later that the system is not completely determined if $\Theta_4 = 0$ or $\Theta_{10} = 0$, so that the theorem really breaks down in these cases.

The invariants (49) together with those formed from them by repetitions of the Jacobian process give a functionally complete system of invariants of system (A). We shall now show this. As we have just seen, the coefficients of the system can be expressed in terms of the invariants $\Theta_3, \Theta_4, --- \vartheta_{10}$ of the system, their derivatives and certain exponentials. Since the seminvariants (11) and (14) are functions of the coefficients, they can also be expressed in terms of the invariants $\Theta_3, \Theta_4 - \vartheta_{10}$, their derivatives and certain exponentials. However, by direct substitution we find that the exponential factors cancel out in the seminvariants, and that the seminvariants are rational algebraic functions of the invariants and their derivatives alone. This fact tells us that any invariant can be expressed as a function of $\Theta_3, \Theta_4, -- \vartheta_{10}$ and of their derivatives. If we write out the system of partial differential equations for the invariants depending on $\Theta_3, \Theta_4, -- \vartheta_{10}$ and their first derivatives, we find that it contains two independent equations and

sixteen variables. Hence there are fifteen relative invariants depending on these quantities. The eight invariants (49) together with the seven obtained by combining Θ_4 with each of the others by the Jacobian process give these fifteen invariants.

The same reasoning tells us that there are seven more invariants depending on the invariants (49) and their first and second derivatives. These can be obtained by combining Θ_4 once more with the other seven of (49) by means of the Jacobian process. It is evident that we can continue in this way and obtain the invariants depending on the invariants (49) and their derivatives up to any order by repeated applications of the Jacobian process.

5. The covariants.

It is easy to verify directly that

(54)
$$E_1 = u_{12} z^2 - u_{21} y^2 + (u_{11} - u_{22}) yz$$

is a semi-covariant of weight 2. Since v_{ik}, t_{ik}, w_{ik} are cogredient with u_{ik} for transformations of the dependent variables, we must have the three additional semi-covariants

(55)
$$\begin{aligned} E_2 &= t_{12} z^2 - t_{21} y^2 + (t_{11} - t_{22}) yz, \\ E_3 &= v_{12} z^2 - v_{21} y^2 + (v_{11} - v_{22}) yz, \\ E_4 &= w_{12} z^2 - w_{21} y^2 + (w_{11} - w_{22}) yz. \end{aligned}$$

Making the transformation

$$y = \alpha\bar{y} + \beta\bar{z}, \quad z = \gamma\bar{y} + \delta\bar{z}, \quad \Delta = \alpha\delta - \beta\gamma \neq 0$$

we find

$$E_1 = \Delta\bar{E}_1,$$

whence

$$\frac{E_1'}{E_1} = \frac{\bar{E}_1'}{\bar{E}_1} + \frac{\Delta'}{\Delta}.$$

Also we have from (15)

$$p_{11} + p_{22} = \bar{p}_{11} + \bar{p}_{22} - \frac{\Delta'}{\Delta}$$

The last two equations show that

(56) $$F_1 = E_1' + (p_{11} + p_{22})E_1$$

is a semi-covariant. In the same way we can obtain three other semi-covariants from equations (55).

Let us put

(57) $$\begin{aligned} \rho &= y' + p_{11}y + p_{12}z, \\ \sigma &= z' + p_{21}y + p_{22}z. \end{aligned}$$

It is easily verified that ρ and σ are cogredient with y and z respectively for transformations of the dependent variables. Consequently we have a semi-covariant

(58) $$P_1 = z\rho - y\sigma.$$

Again, let us put

(59) $$\begin{aligned} \mu &= \rho' + p_{11}\rho + p_{12}\sigma, \\ \nu &= \sigma' + p_{21}\rho + p_{22}\sigma. \end{aligned}$$

Since μ and ν are of the same form in ρ and σ as are ρ and σ in y and z, they are cogredient with ρ and σ and also with y and z . Consequently we have the two futher semi-covariants

(60) $$\begin{aligned} P_2 &= z\mu - y\nu, \\ P_3 &= \sigma\mu - \rho\nu. \end{aligned}$$

We need not consider semi-covariants containing any higher derivatives of y and z than the second. For derivatives of the third and higher orders can be removed by means of the original equations (A). The system of differential equations for the semi-covariants involving the arguments $p_{ik}, p'_{ik}, p''_{ik}, q_{ik}, q'_{ik}, r_{ik}$, and y, y', y'', z, z', z'' is the same as the system for the seminvariants depending on the first twenty-four quantities with the terms containing the partial derivatives with respect to the last six quantities added. This new system contains 16 independent equations and 30 variables. Hence there are 15 seminvariants and relative semi-covariants depending on the arguments

mentioned. Since 9 of the 15 are seminvariants, 6 of them must be semi-covariants. We can take them to be the 6 independent semi-covariants

$$E_1, E_2, F_1, P_1, P_2, P_3.$$

The system of equations for the semi-covariants involving higher derivatives of p_{ik}, q_{ik}, r_{ik} shows that no new independent semi-covariants can occur.

The absolute covariants must be functions of the seminvariants and absolute semi-covariants. Hence the system of differential equations for the covariants will be the same as the system for the invariants except that terms involving derivatives with respect to the five absolute semi-covariants must be added. Thus we see that there are five independent absolute, or six independent relative, covariants.

We can verify directly that

(61) $$E_1, E_2, F, P_1$$

are covariants. From the relations

$$\bar{\mathcal{P}}_1 = \frac{1}{\xi'}\mathcal{P}_1,$$
$$\bar{\Theta}_4 = \frac{1}{(\xi')^4}\Theta_4,$$
$$\bar{\Theta}'_4 = \frac{1}{(\xi')^5}(\Theta'_4 - 4\eta\Theta_4),$$
$$\bar{\mathcal{P}}_2 = \frac{1}{(\xi')^2}(\mathcal{P}_2 + \eta\mathcal{P}_1)$$

we derive at once the covariant

(62) $$4\mathcal{P}_2\Theta_4 + \mathcal{P}_1\Theta'_4.$$

Finally, by integrating the system of differential equations we obtain the sixth independent relative covariant

(63) $$(30I\Theta_4 + \Theta''_4)\mathcal{P}_1 + 9\Theta'_4\mathcal{P}_2 + 36\Theta_4\mathcal{P}_3.$$

We have found the following independent covariants

$$C_1 = \mathcal{P}_1, \quad C_2 = E_1, \quad C_3 = E_2, \quad K_3 = E_1' + (p_{11} + p_{22}) E_1,$$

(64) $$C_6 = 4\mathcal{P}_2 \Theta_4 + \mathcal{P}_1 \Theta_4',$$

$$C_7 = (30 I \Theta_4 + \Theta_4'')\mathcal{P}_1 + 9\Theta_4' \mathcal{P}_2 + 36\Theta_4 \mathcal{P}_3 .$$

<u>all of which are quadratic</u>. <u>The weight of each is indicated by the subscript</u>.

6. <u>Geometrical interpretation</u>. <u>Ruled surfaces in 5 dimensions</u>.

The general theory of differential equations teaches us that the system (A) possesses two <u>general</u> solutions y and z, i.e., two solutions which are analytic in the vicinity of $x = x_0$ if the coefficients are analytic in that vicinity, and which can be made to satisfy also the conditions that y, z, y', z', y'', z'' shall assume arbitrarily prescribed values for $x = x_0$. Furthermore, we easily see that any six pairs of solutions (y_i, z_i), $(i = 1,2,,.6)$, which satisfy the condition that the determinant

(65) $$D = D(y_k'' \; z_k'' \; y_k' \; z_k' \; y_k \; z_k) = \begin{vmatrix} y_1'' & z_1'' & y_1' & z_1' & y_1 & z_1 \\ y_2'' & z_2'' & y_2' & z_2' & y_2 & z_2 \\ y_3'' & z_3'' & y_3' & z_3' & y_3 & z_3 \\ y_4'' & z_4'' & y_4' & z_4' & y_4 & z_4 \\ y_5'' & z_5'' & y_5' & z_5' & y_5 & z_5 \\ y_6'' & z_6'' & y_6' & z_6' & y_6 & z_6 \end{vmatrix}$$

shall not vanish for $x = x_0$, form a fundamental system of solutions of (A). By this statement we mean to say that the general solutions y and z can be expressed in the form

(66) $$y = \sum_{i=1}^{6} c_i y_i, \quad z = \sum_{i=1}^{6} c_i z_i .$$

where $c_1, c_2, \dots c_6$ are arbitrary constants.

If six pairs of functions (y_i, z_i) whose determinant D does not vanish identically are given, we can determine a system of differential equations of the form (A) of which these functions are a

fundamental system of solutions. It is only necessary to solve for p_{ik}, q_{ik}, r_{ik} in the twelve equations

$$(67)\qquad \begin{gathered} y_i''' + 3p_{11}y_i'' + 3p_{12}z_i'' + 3q_{11}y_i' + 3q_{12}z_i' + r_{11}y_i + r_{12}z_i = 0, \\ z_i''' + 3p_{21}y_i'' + 3p_{22}z_i'' + 3q_{21}y_i' + 3q_{22}z_i' + r_{21}y_i + r_{22}z_i = 0, \\ (i = 1, 2, -- 6). \end{gathered}$$

We find for p_{ik} the values

$$(68)\qquad \begin{aligned} D p_{11} &= -\tfrac{1}{3} D(y_i''' z_i'' y_i' z_i' y_i z_i), \\ D p_{12} &= -\tfrac{1}{3} D(y_i'' y_i''' y_i' z_i' y_i z_i), \\ D p_{21} &= -\tfrac{1}{3} D(z_i''' z_i'' y_i' z_i' y_i z_i), \\ D p_{22} &= -\tfrac{1}{3} D(y_i'' z_i''' y_i' z_i' y_i z_i). \end{aligned}$$

The values of q_{ik} and r_{ik} are given by similar expressions.

From equations (68) is obtained at once the relation

$$(69)\qquad D = C e^{-3\int(p_{11}+p_{22})dx},$$

where C is a non-vanishing constant.

If the general solutions of system (A) undergo the transformation

$$(70)\qquad \eta = \alpha y + \beta z, \quad \zeta = \gamma y + \delta z,$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions of x, then η and ζ will be the general solutions of a new system of equations of the same form as (A). If also we put

$$(71)\qquad \eta_i = \alpha y_i + \beta z_i, \quad \zeta_i = \gamma y_i + \delta z_i, \quad (i = 1, 2, -- 6),$$

the six pairs of functions (η_i, ζ_i) will form a fundamental system of solutions of the new system. The general solutions will be

$$(72)\qquad \eta = \sum_{i=1}^{6} c_i \eta_i, \quad \zeta = \sum_{i=1}^{6} c_i \zeta_i.$$

From these considerations it is evident that six pairs of solutions which form a fundamental system are transformed cogrediently with eachother and with the pair of general solutions.

Let us now assume that (A) has been integrated, so that we have

$(y_1, y_2, y_3, y_4, y_5, y_6)$ and $(z_1, z_2, z_3, z_4, z_5, z_6)$ expressed as functions of x. We can interpret these two sets of quantities as the homogeneous coordinates of two points P_y and P_z in five- dimensional space. Then as x varies these two points describe two curves C_y and C_z which we shall call integral curves. The points on these two curves can be put into a definite correspondence by regarding those as corresponding points which belong to the same value of x.

However, this correspondence must not be such that the determinant D defined by equation (65) is zero. This is equivalent to saying that it must not be possible to find six functions of x, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ such that the equations

(73) $$\lambda_1 y_i + \lambda_2 y_i' + \lambda_3 y_i'' + \lambda_4 z_i + \lambda_5 z_i' + \lambda_6 z_i'' = 0, \quad (i = 1, 2, \dots 6),$$

are satisfied. These equations can be written in the form

$$\lambda_1 y_i + \lambda_2 y_i' + \lambda_3 y_i'' = -(\lambda_4 z_i + \lambda_5 z_i' + \lambda_6 z_i''),$$

The left member represents a point in the osculating plane* of C_y at P_y while the right member represents a point in the osculating plane of C_z at P_z. If equations (73) hold, these two points coincide for all values of x. Therefore, <u>the necessary and sufficient condition that C_y and C_z may be the integral curves of a system of form</u> (A) <u>is that the osculating planes to these two curves at corresponding points shall not in general intersect. This implies further that the osculating planes at corresponding points must not lie in the</u>

* For the definition of an osculating space of n dimensions of C_y and the proof that the coordinates of n independent points of it are given by $y_i, y_i', \dots y_i^{(n)}$, see Wilczynski, loc. cit., p. 54.

same space of four dimensions.

Let us join corresponding points P_y and P_z by a straight line L_{yz}. As x takes on all values, the line L_{yz} will generate a ruled surface. Now the transformation (71) converts P_y and P_z into two points P_η and P_ζ on the line L_{yz}. Since $\alpha, \beta, \gamma, \delta$ are arbitrary functions of x, we can convert the curves C_y and C_z into any two curves C_η and C_ζ which intersect every generator. It is evident that the lines joining corresponding points on C_η and C_ζ remain generators of the surface. Moreover, the most general transformation of the independent variable simply changes the parametric representation of the curves C_y and C_z. Thus we have the following theorem.

To every fundamental system of solutions of a system of differential equations of the form (A) there corresponds a ruled surface in five-dimensional space. This ruled surface is unchanged by transformations of the form

$$\eta = \alpha y + \beta z, \quad \zeta = \gamma y + \delta z, \quad \xi = f(x),$$

where $\alpha, \beta, \gamma, \delta$ and f are arbitrary functions of x.

However, this integrating ruled surface must not be such that the osculating planes of C_y and C_z at corresponding points intersect. This means that three consecutive generators of the surface must not lie in the same space of four dimensions. It is evident at once that the surface cannot have two distinct sets of generators. In fact, it is impossible to construct a straight line intersecting three consecutive generators. A ruled surface in five dimensions for which any three consecutive generators lie in the same space of four dimensions we shall call a pseudo- developable. These pseudo- developables are excluded from our theory.

There may be particular values of x which make the determinant

D equal to zero. Along the generator determined by such a value of x the osculating planes to the two integral curves intersect. We shall call a generator of this kind a <u>pseudo-torsal generator</u>.

Now we have seen that

$$D = C e^{-3\int(p_{11}+p_{22})dx}$$

For a value of x which gives a pseudo-torsal generator, one or both of the quantities $\int p_{11}dx$, $\int p_{22}dx$ must be infinite. Therefore, for such a value of x the transformation

$$y = \eta e^{-\int p_{11}dx}, \quad z = \zeta e^{-\int p_{22}dx},$$

which gives a system of form (A) with $p_{11} = p_{22} = 0$, is not permissible.

The integrating surface of (A) has been defined by a particular fundamental system of solutions. But any other fundamental system $(\bar{y}_k, \bar{z}_k)$ is given by the expressions

$$\bar{y}_k = \sum_{i=1}^{6} c_{ki}\, y_i, \quad \bar{z}_k = \sum_{i=1}^{6} c_{ki}\, z_i, \quad (k = 1, 2, \cdots 6).$$

Therefore, any projective transformation of the original surface is also an integrating surface of (A).

Suppose now that the integrating surfaces of two systems of the form (A) coincide. Any pair of curves on the surface can be transformed into any other pair by (70), and the independent variables are expressible in terms of each other since to each generator there corresponds a value of each variable. As the surface has but one set of generators, it is evident that the two systems of form (A) can be transformed into each other.

If we have given any ruled surface in five-dimensional space which is not a pseudo-developable, it is easy to find a system of form (A) which defines it. It is only necessary to trace upon the surface

two curves which are not generators and to express the coordinates of their points as functions of a parameter x in such a way that points given by the same value of x lie on the same generator. The coefficients of the required system are at once given by equations (68) and the corresponding equations for q_{ik} and r_{ik}.

We have now shown that *there belongs to every system of form* (A) *a ruled surface in five-dimensional space which is not a pseudo-developable and, conversely, every such surface may be defined by a system of form (A)*. Therefore, the projective properties of the integrating ruled surface can be obtained by studying the system (A). Every invariant equation or system of equations of (A) expresses a projective property of the integrating surface and, conversely, every projective property of the surface can be expressed by an invariant equation, or system of equations.

It has been seen that, if the invariants $\Theta_3, \Theta_4, \ldots \Theta_{10}$ are given as functions of x, provided that Θ_4 and Θ_{10} are not equal to zero, a system of equations of form (A) having these functions for its invariants is determined uniquely except for transformations of the dependent variables. This fundamental theorem can now be stated in the following form.

If the invariants $\Theta_3, \Theta_4, \ldots \Theta_{10}$ *are given as functions of* x, *provided that* Θ_4 *and* Θ_{10} *are not equal to zero, they determine a ruled surface in five-dimensional space uniquely except for projective transformations.*

7. Dualistic interpretation.

Let us consider a ruled surface S which is defined by a system of form (A). Just as with surfaces in ordinary space, a line is said

to be tangent to S at a point P if it is tangent at P to some curve on S. We shall now define the tangent spaces of two, three, and four dimensions. Through-out this paper any linear space of r dimensions will be called an r-space, and any r-dimensional continuum, which is not necessarily linear, will be called an r-spread.

The tangent line at P_y to an integral curve C_y together with the generator through P_y determine a 2-space, or plane. Every line tangent to S at P_y lies in this plane. We shall now show this. The coordinates of a point $P_{\bar{y}}$ of any curve $C_{\bar{y}}$ on S are given by

$$\bar{y}_k = \alpha y_k + \beta z_k \quad , \quad (k = 1, 2, \dots 6),$$

where y_k and z_k are the coordinates of the points on the same generator of the two integral curves, and α and β are arbitrary functions of x. The tangent line to $C_{\bar{y}}$ at $P_{\bar{y}}$ is determined by the points $P_{\bar{y}}$ and $P'_{\bar{y}}$, where $P'_{\bar{y}}$ is the point whose coordinates are

$$\bar{y}'_k = \alpha y'_k + \beta z'_k + \alpha' y_k + \beta' z_k .$$

If $P_{\bar{y}}$ is made to coincide with P_y, where P_y lies on the generator corresponding to the value $x = x_0$, we must have $\beta = 0$ for $x = x_0$. The line tangent to $C_{\bar{y}}$ at $P_{\bar{y}} = P_y$ is now determined by two points whose coordinates are given by

$$\bar{y}_k = \alpha y_k ,$$
$$\bar{y}'_k = \alpha y'_k + \alpha' y_k + \beta' z_k .$$

But the plane defined above contains the points P_y, P'_y, P_z, and contains therefore, the line tangent to $C_{\bar{y}}$ at P_y. Thus, there is at each point of S a plane containing all the tangent lines at the point. <u>We shall call it the tangent plane at the point.</u>

The tangent planes at all points of a generator L_{yz} are evidently contained in a 3-space determined by L_{yz} and its next consecutive generator. <u>We shall speak of this 3-space as the tangent 3-space</u>

<u>along the generator</u> L_{yz}.

The osculating plane at P of any curve on the surface through P, together with the tangent 3-space along the generator through P, determine a 4-space, and this 4-space contains the osculating planes at P of all the curves through P. This fact can be proved by an extension of the method used in proving that the tangent plane at a point contains all the tangent lines at the point. <u>We shall call this 4-space the tangent 4-space at</u> P.

The principle of duality teaches us that, in 5-dimensional space, to a point corresponds a 4-space, to a line corresponds a 3-space, and to a plane corresponds again a plane.

We wish to think of our ruled surface not merely as a point locus, but as capable of generation by means of lines, planes, 3-spaces, and 4-spaces as well. Moreover, we wish to do this in such a way that the combined point, line, 3-space, and 4-space locus may be converted into another one of the same kind by a dualistic transformation. In order to do this, let us associate with any point P_y of S the tangent 4-space at P_y. If P_y and P_z are two corresponding points of the integral curves C_y and C_z, the tangent 4-space at P_y is determined by the points $P_y, P_y', P_y'', P_z, P_z'$. Hence, in accordance with Grassmann's definition, its coordinates u_k are proportional to the minors of z_k'' in D. Likewise the coordinates v_k of the tangent 4-space at P_z are proportional to the minors of y_k'' in D. We shall indicate these facts by writing

$$u = |y''y'z'yz|, \qquad v = |z''y'z'yz|.$$

The notation in similar cases will explain itself.

In order to attain the self-dual character which we wish our combined point, line, 3-space, 4-space locus to have, the line determin-

ed by y and z must correspond to the 3-space common to u and v, i.e., to the 3-space $|y'z'yz|$, which is the tangent 3-space of S along L_{yz}. Also, the plane determined by y', y, z must correspond to the plane common to u', u, v. From the definition of u we have

(74) $$u' = -(y''z''y'y\,z) - 3p_{11}u - 3p_{12}v,$$

so that the tangent plane corresponds to the plane $|y'y\,z|$, i.e., to itself. Furthermore, L_{yz} is the line common to u', v', u, v, and P_y is the point common to u'', u', v', u, v. Thus, we have established between the elements of the ruled surface a self-dual one-to-one correspondence in which each point of the surface corresponds to its tangent 4-space, each generator to its tangent 3-space, and each tangent plane to itself.

It has already been noted that the coordinates u_k and v_k of the tangent 4-spaces at P_y and P_z respectively are proportional to certain minors in D. Let us now make the transformation

(75) $$y_k = \alpha\bar{y}_k + \beta\bar{z}_k, \quad z = \gamma\bar{y}_k + \delta\bar{z}_k, \quad (k = 1, 2, \cdots 6).$$

By some easy calculations it is proved that $\underline{u_k}$ <u>and</u> $\underline{v_k}$ <u>are transformed according to the equations</u>

(76) $$u_k = \Delta^2(\alpha\bar{u}_k + \beta\bar{v}_k), \quad v_k = \Delta^2(\gamma\bar{u}_k + \delta\bar{v}_k),$$
$$(k = 1, 2, \cdots 6),$$

<u>where</u>

$$\Delta = \alpha\delta - \beta\gamma,$$

<u>a direct generalization of Chasles' correlation for ordinary ruled surfaces.</u>

By direct calculation, the value of D after the transformation (75) is found to be

(77) $$D = \Delta^3\bar{D}.$$

Consequently, the quantities U_k and V_k, where

(78) $$U_k = \frac{u_k}{D^{2/3}} \qquad V_k = \frac{v_k}{D^{2/3}},$$

are absolutely cogredient with y_k and z_k .

The value of the determinant formed from u_k and v_k in the same way that D was formed from y_k and z_k, is D^5 and it cannot, therefore, vanish identically. Hence we can regard u_k and v_k as a fundamental system of solutions of a pair of equations of the same form as (A). These equations are easily found by differentiating u_k and v_k three times and making the necessary eliminations.

However, we shall have more occasion to use the similar system for which U_k and V_k are a fundamental system of solutions. Since we have from (69) the relation

$$D = C e^{-3\int(p_{11}+p_{22})dx},$$

the new system may be obtained by making the transformation

(79) $$u = U e^{-2\int(p_{11}+p_{22})dx}, \quad v = V e^{-2\int(p_{11}+p_{22})dx}.$$

It is found to be

$$\begin{aligned} &U''' + 3p_{11}U'' + 3p_{12}V'' + 3(q_{11}+u_{11}-u_{22})U' + 3(q_{12}+2u_{12})V' \\ &\quad + \{r_{11} + 3p_{11}(u_{11}-u_{22}) + 6p_{21}u_{12} + \tfrac{3}{2}(v_{11}-v_{22}) - \tfrac{1}{2}(t_{11}+t_{22})\}U \\ &\quad + \{r_{12} + 3p_{12}(u_{11}-u_{22}) + 6p_{22}u_{12} + 3v_{12}\}V = 0, \end{aligned}$$

(80) $$\begin{aligned} &V''' + 3p_{21}U'' + 3p_{22}V'' + 3(q_{21}+2u_{21})U' + 3(q_{22}-u_{11}+u_{22})V' \\ &\quad + \{r_{21} - 3p_{21}(u_{11}-u_{22}) + 6p_{11}u_{21} + 3v_{21}\}U \\ &\quad + \{r_{22} - 3p_{22}(u_{11}-u_{22}) + 6p_{12}u_{21} - \tfrac{3}{2}(v_{11}-v_{22}) - \tfrac{1}{2}(t_{11}+t_{22})\}V = 0, \end{aligned}$$

where u_{ik}, v_{ik}, t_{ik} are the quantities so designated previously. <u>We shall speak of the system (80) as the adjoint of</u> (A).

Not only is (80) the adjoint of (A), but also (A) is the adjoint of (80). To show this, let us form the quantities $U_{ik}, V_{ik}, T_{ik}, W_{ik}$

from (80) in the same way that we have formed $u_{ik}, v_{ik}, t_{ik}, w_{ik}$ from (A). We find

$$
(81)\quad
\begin{array}{llll}
U_{11} = u_{22}, & U_{12} = -u_{12}, & U_{21} = -u_{21}, & U_{22} = u_{11}, \\
V_{11} = v_{22}, & V_{12} = -v_{12}, & V_{21} = -v_{21}, & V_{22} = v_{11}, \\
T_{11} = -t_{22}, & T_{12} = t_{12}, & T_{21} = t_{21}, & T_{22} = -t_{11}, \\
W_{11} = w_{22}, & W_{12} = -w_{12}, & W_{21} = -w_{21}, & W_{22} = w_{11}.
\end{array}
$$

If we denote the coefficients of (80) by P_{ik}, Q_{ik}, R_{ik}, we have

$$
(82)\quad
\begin{aligned}
&P_{ik} = p_{ik}, \\
&Q_{11} = q_{11} + u_{11} - u_{22}, \qquad Q_{12} = q_{12} + 2u_{12}, \\
&Q_{21} = q_{21} + 2u_{21}, \qquad Q_{22} = q_{22} - u_{11} + u_{22}, \\
&R_{11} = r_{11} + 3p_{11}(u_{11} - u_{22}) + 6p_{21}u_{12} + \tfrac{3}{2}(v_{11} - v_{22}) - \tfrac{1}{2}(t_{11} + t_{22}), \\
&R_{12} = r_{12} + 3p_{12}(u_{11} - u_{22}) + 6p_{22}u_{12} + 3v_{12}, \\
&R_{21} = r_{21} - 3p_{21}(u_{11} - u_{22}) + 6p_{11}u_{21} + 3v_{21}, \\
&R_{22} = r_{22} - 3p_{22}(u_{11} - u_{22}) + 6p_{12}u_{21} - \tfrac{3}{2}(v_{11} - v_{22}) - \tfrac{1}{2}(t_{11} + t_{22}).
\end{aligned}
$$

From these equations and equations (81) we have

$$
(83)\quad
\begin{aligned}
&p_{ik} = P_{ik}, \\
&q_{11} = Q_{11} + U_{11} - U_{22}, \qquad q_{12} = Q_{12} + 2U_{12}, \\
&q_{21} = Q_{21} + 2U_{21}, \qquad q_{22} = Q_{22} - U_{11} + U_{22}, \\
&r_{11} = R_{11} + 3P_{11}(U_{11} - U_{22}) + 6P_{21}U_{12} + \tfrac{3}{2}(V_{11} - V_{22}) - \tfrac{1}{2}(T_{11} + T_{22}), \\
&r_{12} = R_{12} + 3P_{12}(U_{11} - U_{22}) + 6P_{22}U_{12} + 3V_{12}, \\
&r_{21} = R_{21} - 3P_{21}(U_{11} - U_{22}) + 6P_{11}U_{21} + 3V_{21}, \\
&r_{22} = R_{22} - 3P_{22}(U_{11} - U_{22}) + 6P_{12}U_{21} - \tfrac{3}{2}(V_{11} - V_{22}) - \tfrac{1}{2}(T_{11} + T_{22}).
\end{aligned}
$$

These equations show that p_{ik}, q_{ik}, r_{ik} are formed from P_{ik}, Q_{ik}, R_{ik} in the same way that P_{ik}, Q_{ik}, R_{ik} are formed from p_{ik}, q_{ik}, r_{ik}, i.e., the relation between equations (A) and (80) is a reciprocal one.

Equations (82) show that the adjoint of (A) coincides with (A)

<u>if and only if</u>

(84) $$u_{11} - u_{22} = u_{12} = u_{21} = t_{11} + t_{22} = 0.$$

Suppose that these conditions are satisfied. Then, since (y_k, z_k) is a fundamental system of solutions for the system (80) as well as the system (A), it must be possible to express U_k and V_k in terms of y_k and z_k by means of equations of the form

(85) $$U_i = \sum_{k=1}^{6} c_{ik} y_k, \quad V_i = \sum_{k=1}^{6} c_{ik} z_k,$$

where c_{ik} are constants. From the definition of U_k and V_k we have

(86) $$\Sigma U_i y_i = 0, \ \Sigma V_i z_i = 0, \ \Sigma U_i z_i = 0, \ \Sigma V_i y_i = 0,$$

where the summation is always from 1 to 6. If the values of U_i and V_i from (85) are substituted in (86), it becomes evident that <u>the system (A) and its adjoint coincide only if the entire surface</u> S <u>lies in a quadratic 4-spread</u>.

We shall now show the geometrical significance of the semi-canonical form. In addition to the relations (86) we have from the definition of U_k and V_k

(87) $$\begin{aligned} &\Sigma U_i z_i'' = -D^{\frac{1}{3}}, && \Sigma V_i y_i'' = D^{\frac{1}{3}}, \\ &\Sigma U_i y_i'' = 0, && \Sigma V_i z_i'' = 0, \\ &\Sigma U_i z_i' = 0, && \Sigma V_i y_i' = 0, \\ &\Sigma U_i y_i' = 0, && \Sigma V_i z_i' = 0. \end{aligned}$$

If we multiply equations (67) by U_k and V_k successively and add, we find by using the relations (86) and (87) that

(88) $$\begin{aligned} &\Sigma U_i y_i''' = 3 p_{12} D^{\frac{1}{3}}, && \Sigma V_i z_i''' = -3 p_{21} D^{\frac{1}{3}}, \\ &\Sigma U_i z_i''' = 3 p_{22} D^{\frac{1}{3}}, && \Sigma V_i y_i''' = -3 p_{11} D^{\frac{1}{3}}. \end{aligned}$$

Now we have seen that the system (A) can always be transformed into one for which $p_{12} = p_{21} = 0$. Let us assume that this transformation has been made. Then equations (86), (87), and (88) give the relations

$$
(89) \qquad \begin{aligned}
&\Sigma u_i y_i = 0, \quad \Sigma u_i y_i' = 0, \quad \Sigma u_i y_i'' = 0, \quad \Sigma u_i y_i''' = 0, \\
&\Sigma v_i z_i = 0, \quad \Sigma v_i z_i' = 0, \quad \Sigma v_i z_i'' = 0, \quad \Sigma v_i z_i''' = 0.
\end{aligned}
$$

Thus, the integral curve C_y is such that its osculating 3-space at P_y is contained in the tangent 4-space of the surface at P_y. In other words, the tangent 4-space at a point on C_y has four consecutive points in common with C_y. Likewise the tangent 4-space at a point on C_z has four consecutive points in common with C_z. Because of their close analogy to asymptotic curves on ordinary ruled surfaces, we shall call the curves C_y and C_z, when determined so that the conditions $p_{12} = p_{21} = 0$ are satisfied, <u>pseudo-asymptotic curves</u>.

The system for which $p_{12} = p_{21} = 0$ can be converted into another for which also $p_{11} = p_{22} = 0$ by the transformation

$$y = \eta e^{-\int p_{11} dx}, \quad z = \zeta e^{-\int p_{22} dx}.$$

Such a transformation evidently has no geometrical significance. Hence, if the system (A) is in its semi-canonical form the integral curves on S are pseudo-asymptotic curves.

The most general transformation leaving the semi-canonical form invariant is

$$\xi = \xi(x), \quad \bar{y} = \xi'(ay + bz), \quad \bar{z} = \xi'(cy + dz),$$

where ξ is an arbitrary function and a, b, c, d are arbitrary constants. Therefore, <u>there exists upon the ruled surface S a single infinity of pseudo-asymptotic curves</u>. <u>Moreover, the double ratio of</u>

the four points in which any generator intersects four fixed pseudo-asymtotic curves is constant. This is an extension to ruled surfaces in 5-dimensional space of Serret's well known theorem for ruled surfaces in ordinary space.

If we differentiate equations (67), multiply the result by U_k and V_k successively and add, we find

$$(90)\qquad \begin{aligned} \sum U_i y_i^{(4)} &= 3D^{1/3}\{p_{12}' + q_{12} - 3(p_{11}+p_{22})p_{12}\}, \\ \sum V_i z_i^{(4)} &= -3D^{1/3}\{p_{21}' + q_{21} - 3(p_{11}+p_{22})p_{21}\}. \end{aligned}$$

If the conditions

$$(91)\qquad p_{12} = p_{21} = u_{12} = u_{21} = 0$$

are satisfied, we have in addition to (89) the relations

$$(92)\qquad \sum U_i y_i^{(4)} = 0, \quad \sum V_i z_i^{(4)} = 0.$$

Therefore, under the conditions (91) the osculating 4-space at P_y of C_y coincides with the tangent 4-space at P_y of S. We shall later find another interpretation for the conditions (91).

We may re-interpret the solutions U_k and V_k of the adjoint of (A) as the homogeneous coordinates of a point in 5-dimensional space. Then the adjoined system determines a second ruled surface dual to the original surface S.

Let us denote the invariants of the adjoint of (A) by $\bar\Theta_3, \bar\Theta_4$, etc. From (81) we find

$$(93)\qquad \begin{aligned} \bar\Theta_3 &= -\Theta_3, & \bar\Theta_8 &= \Theta_8, \\ \bar\Theta_4 &= \Theta_4, & \bar\Theta_9 &= -\Theta_9, \\ \bar\Theta_5 &= -\Theta_5, & \bar\Theta_{10} &= \Theta_{10}, \\ \bar\Theta_6 &= \Theta_6, & \bar\vartheta_{10} &= -\vartheta_{10}. \end{aligned}$$

Therefore, (A) and its adjoint can be transformed into each other if

and only if

(94) $$\Theta_3 = \Theta_5 = \Theta_9 = \vartheta_{10} = 0$$

When these conditions are satisfied, there exists a dualistic transformation which converts the ruled surface into itself generator for generator. We shall say that the surface S is *identically self-dual.*

If the conditions (84) are satisfied there exists a dualistic transformation which converts each point of the surface into its tangent 4-space and each tangent 4-space into its point of contact, or as we might say, the surface is identically self-dual point for point.

8. The derivative surfaces.

Let us return to the consideration of equations (57). If we substitute $y = y_k$, $z = z_k$, $(k=1,2,\ldots 6)$, in these expressions, we find

(95) $$\begin{aligned} \rho_k &= y_k' + p_{11} y_k + p_{12} z_k, \\ \sigma_k &= z_k' + p_{21} y_k + p_{22} z_k, \end{aligned} \qquad (k = 1, 2, \ldots 6).$$

Evidently the ρ_k's are the homogeneous coordinates of a point P_ρ in the tangent plane of S at P_y, while the σ_k's are the coordinates of a point P_σ in the tangent plane of S at P_z.

It has been shown that ρ and σ are cogredient with y and z for transformations of the form

$$y = \alpha\bar{y} + \beta\bar{z}, \quad z = \gamma\bar{y} + \delta\bar{z}.$$

Therefore, when P_y and P_z are transformed into two points $P_{\bar{y}}$ and $P_{\bar{z}}$ of the generator L_{yz}, P_ρ and P_σ are transformed into two points $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$ of $L_{\bar{\rho}\bar{\sigma}}$ which lie in the tangent plane of S at $P_{\bar{y}}$ and $P_{\bar{z}}$ respectively. Thus, by means of equations (95), there is made to correspond to each generator L_{yz} of S a line $L_{\rho\sigma}$ in the 3-space tangent to S along L_{yz}.

Let us now assume that (A) has been reduced to its semi-canonical form, so that the integral curves are pseudo-asymptotic curves. Then equations (95) become

$$\rho_k = y'_k, \quad \sigma_k = z'_k.$$

The points P_ρ and P_σ not only lie in the tangent planes of S at P_y and P_z respectively but also on the tangent lines to C_y and C_z at P_y and P_z respectively. Thus, there is established between the points of L_{yz} and $L_{\rho\sigma}$ a one-to-one correspondence in which the point P' of $L_{\rho\sigma}$ corresponding to any point P of L_{yz} lies on the line t tangent at P to the pseudo- asymptotic curve through P. As P moves along L_{yz}, the line t describes a quadric surface Q in the 3-space tangent along L_{yz}. For, t always intersects the generator L_{yz}, the next consecutive generator, and the line $L_{\rho\sigma}$.

Since a transformation

$$\xi = \xi(x)$$

of the independent variable converts ρ and σ into $\bar{\rho}$ and $\bar{\sigma}$ where

$$\bar{\rho} = \frac{1}{\xi'}(\rho + \eta y), \quad \bar{\sigma} = \frac{1}{\xi'}(\sigma + \eta z), \quad \eta = \frac{\xi''}{\xi'},$$

the position of $L_{\rho\sigma}$ on Q is changed. In fact, by giving η the proper value, we can make $L_{\rho\sigma}$ coincide with any generator on Q of the same kind as $L_{\rho\sigma}$.

Let us now construct the line $L_{\rho\sigma}$ corresponding to each generator of S. We obtain a new ruled surface S', which we shall call <u>the first derivative of the first kind of S with respect to x</u>. When the independent variable is transformed, S' changes unless $\eta = 0$ for the transformation, i.e., unless the transformation is linear and integral of the form

$$\xi = ax + b,$$

where a and b are constants.

Equations (59) define two quantities μ and ν which are formed from ρ and σ in the same way as ~~that~~ ρ and σ are formed from y and z. We have, as before

$$(96) \qquad \begin{aligned} \mu_k &= \rho_k' + p_{11}\rho_k + p_{12}\sigma_k, \\ \nu_k &= \sigma_k' + p_{21}\rho_k + p_{22}\sigma_k. \end{aligned}$$

It is clear that P_μ is a point in the tangent plane of S' at P_ρ and P_ν is a point in the tangent plane of S' at P_σ. If we substitute the values of ρ_k and σ_k into (96), we have

$$(97) \qquad \begin{aligned} \mu_k &= y_k'' + 2p_{11}y_k' + 2p_{12}z_k' + (u_{11}+q_{11})y_k + (u_{12}+q_{12})z_k, \\ \nu_k &= z_k'' + 2p_{21}y_k' + 2p_{22}z_k' + (u_{21}+q_{21})y_k + (u_{22}+q_{22})z_k. \end{aligned}$$

These equations show that P_μ lies in the tangent 4-space of S at P_y and P_ν in the tangent 4-space at P_z. If equations (A) are reduced to their semi-canonical form, equations (96) and (97) become

$$\mu_k = \rho_k', \qquad \nu_k = \sigma_k',$$

and

$$\mu_k = y_k'', \qquad \nu_k = z_k''.$$

The point P_μ lies in the line tangent to C_ρ at P_ρ and this tangent line, in turn, lies in the osculating plane of C_y at P_y. The point P_ν is similarly located.

Any transformation of the dependent variables transforms μ and ν cogrediently with ρ and σ and also cogrediently with y and z. Therefore, such a transformation changes P_μ and P_ν into two points $P_{\bar\mu}$ and $P_{\bar\nu}$ of the line $L_{\mu\nu}$ at the same time that it transformes P_ρ and P_σ into two points $P_{\bar\rho}$ and $P_{\bar\sigma}$ of the line $L_{\rho\sigma}$. Thus, to each generator $L_{\rho\sigma}$ of S' there corresponds a line $L_{\mu\nu}$, and we have another ruled surface S_1' associated with S'. We shall speak of this new surface as <u>the derivative of the second kind of S with respect to x</u>.

It is evident that S_1' changes when the independent variable is transformed.

If we omit the subscripts in (97) and differentiate, we find a pair of equations from which can be obtained by the elimination of y''', y'', z'' from the first and of z''', y'', z'' from the second, the equations

$$
(98)\quad
\begin{aligned}
\mu' + p_{11}\mu + p_{12}\nu &= 3u_{11}y' + 3u_{12}z' + (3u_{11}p_{11} + 3u_{12}p_{21} + \tfrac{3}{2}v_{11} - \tfrac{1}{2}t_{11})y \\
&\quad + (3u_{11}p_{12} + 3u_{12}p_{22} + \tfrac{3}{2}v_{12} - \tfrac{1}{2}t_{12})z, \\
\nu' + p_{21}\mu + p_{22}\nu &= 3u_{21}y' + 3u_{22}z' + (3u_{21}p_{11} + 3u_{22}p_{21} + \tfrac{3}{2}v_{21} - \tfrac{1}{2}t_{21})y \\
&\quad + (3u_{21}p_{12} + 3u_{22}p_{22} + \tfrac{3}{2}v_{22} - \tfrac{1}{2}t_{22})z.
\end{aligned}
$$

It is easily verified that these equations are unchanged by a transformation of the independent variable. They give the points of intersection of the tangent 3-space along L_{yz} with the tangent planes of S_1' at P_μ and P_ν. They also show that the tangent planes of S at P_y and P_z meet the tangent planes of S_1' at P_μ and P_ν respectively if and only if the conditions

$$u_{12} = u_{21} = 0$$

are satisfied. The left members of equations (98) are formed from μ and ν in the same way that μ and ν are formed from ρ and σ. They may be regarded as defining a new ruled surface S_2' associated with S_1'. Then equations (98) show that the tangent planes of S at P_y and P_z contain the corresponding points of S_2' if and only if $u_{12} = u_{21} = 0$.

It has been seen that it is always possible to reduce the system (A) to another for which $u_{12} = u_{21} = 0$ provided that $\Theta_4 \neq 0$. In fact, if we factor the covariant E_1 and denote its factors by η and ζ, we have at once a transformation which converts (A) into the desired form. The factors are

$$\eta = \frac{u_{11} - u_{22} - \sqrt{\Theta_4}}{2} y + u_{12} z,$$

(99) $$\zeta = \frac{u_{11} - u_{22} + \sqrt{\Theta_4}}{2} y + u_{12} z.$$

Since the most general transformation of the dependent variables which preserves the conditions $u_{12} = u_{21} = 0$ may be obtained from

$$y = \alpha\eta, \quad z = \delta\zeta,$$
$$y = \zeta, \quad z = \eta,$$

the curves C_y and C_z are uniquely determined. Thus, there exists upon S two curves, obtained by factoring the covariant E_1, such that the tangent plane at any point of either of them always contains the corresponding point of the surface S_2'. Of course these two curves, in general, are parts of the same irreducible curve which intersects each generator in two points. Equations (99) show that the two parts are distinct if $\Theta_4 \neq 0$.

It should be noticed here that the factors of the covariant E_2 determine a system of equations for which the conditions $t_{12} = t_{21} = 0$ are satisfied. These conditions completely determine two curves on S which are distinct if $\Theta_3^2 - 4\Theta_6 \neq 0$.

In this paper we shall denote by C_u the curve determined by the conditions $u_{12} = u_{21} = 0$, and by C_t the curve determined by the conditions $t_{12} = t_{21} = 0$.

Suppose that a branch of C_u coincides with a pseudo-asymptotic curve. We can take it to be the integral curve C_y so that we have $u_{12} = p_{12} = 0$. Then the first equation of (98) becomes

$$\mu' + p_{11}\mu = 3u_{11}y' + (3u_{11}p_{11} + \tfrac{3}{2}v_{11} - \tfrac{1}{2}t_{11})y - \tfrac{1}{2}t_{12}z.$$

Hence the tangent plane to S at P_y meets the line tangent to C_μ at the point on S_1' corresponding to P_y.

The necessary and sufficient condition that a branch of C_u coincide with a pseudo-asymptotic curve is $\Theta_{10} = 0$ provided that $\Theta_4 \neq 0$. For, suppose that system (A) is in the form for which $u_{12} = u_{21} = 0$. Then we have

$$\Theta_{10} = (u_{11} - u_{22})^2 p_{12} p_{21} = \Theta_4 p_{12} p_{21}.$$

Therefore, $\Theta_{10} = 0$ if and only if $p_{12} = 0$ or $p_{21} = 0$.

If the conditions

$$p_{12} = u_{12} = t_{12} = 0$$

are satisfied, the first equation of (98) becomes

$$\mu' + p_{11}\mu = 3u_{11}y' + (3p_{11}u_{11} + \tfrac{3}{2}v_{11} - \tfrac{1}{2}t_{11})y.$$

Hence the tangent line to C_y at P_y meets the tangent line to C_μ at P_μ. Moreover, under these conditions the quantities q_{12} and r_{12} also vanish and the first equation of (A) becomes

$$y''' + 3p_{11}y'' + 3q_{11}y' + r_{11}y = 0.$$

As this equation has only three independent solutions, there must be three linear homogeneous relations between the six solutions $y_1, \ldots y_6$. In other words, <u>the curve C_y lies in a plane</u>.

Let us assume that the two branches of C_u do not coincide with each other and also that they do not both coincide with pseudo-asymptotic curves. Then the necessary and sufficient condition that a branch of C_u, a branch of C_t, and a pseudo-asymptotic curve coincide is

$$\Theta_{10} = \vartheta_{10} = 0.$$

For, we can take the integral curves to be the two branches of C_u. Then we have

$$\Theta_{10} = \Theta_4 p_{12} p_{21},$$
$$\vartheta_{10} = 2\Theta_4^{3/2}(p_{21}t_{12} - p_{12}t_{21})$$

It is evident that $\Theta_{10} = \vartheta_{10} = 0$ if and only if $p_{12} = t_{12} = 0$ or $p_{21} = t_{21} = 0$.

We shall now show that <u>S is not determined by its invariants</u> $\Theta_3, \Theta_4, \ldots \vartheta_{10}$ <u>when</u> $\Theta_4 = 0$ <u>or</u> $\Theta_{10} = 0$, i.e.,<u>when a branch of C_u coincides with a pseudo-asymptotic curve or when the two branches of C_u coincide</u>. Let us first take $\Theta_{10} = 0$, $\Theta_4 \neq 0$. We can reduce the system (A) to one for which $p_{11} = p_{22} = u_{12} = u_{21} = 0$. Then we have either $p_{12} = 0$ or $p_{21} = 0$. We shall take $p_{12} = 0$ and then q_{12} must also vanish. The non-vanishing fundamental invariants are not sufficient for the determination of p_{21}, so that it may be taken as an arbitrary function of x. The most general transformation of the dependent variables which leaves all of these conditions unchanged contains only arbitrary constants and cannot, therefore, remove an arbitrary function. Moreover, the arbitrary function cannot be removed by any definite choice of the independent variable.

Let us next assume that the two branches of C_u coincide, so that we have $\Theta_4 = 0$. We may reduce the system (A) to one for which

$$u_{12} = u_{11} - u_{22} = p_{11} = p_{22} = p_{21} = 0.$$

Moreover, we may choose the independent variable so as to make

$$u_{11} + u_{22} = 0.$$

The coefficients of the system when determined by the fundamental invariants which do not vanish under these conditions contain two arbitrary functions of x. Since the most general transformation leaving the above conditions invariant contains only arbitrary constants, the two arbitrary functions cannot be removed.

9. <u>The differential equations for S'</u>.

If we substitute for μ, μ', ν, ν' in (98) their values in terms of $\rho, \rho', \sigma, \sigma'$, and eliminate y' and z' by means of the equations defining ρ and σ, we have

$$(100)\qquad \begin{aligned} \mathcal{R} &= \rho'' + 2p_{11}\rho' + 2p_{12}\sigma' + (q_{11} - 2u_{11})\rho + (q_{12} - 2u_{12})\sigma = s_{11}y + s_{12}z, \\ \mathcal{S} &= \sigma'' + 2p_{21}\rho' + 2p_{22}\sigma' + (q_{21} - 2u_{21})\rho + (q_{22} - 2u_{22})\sigma = s_{21}y + s_{22}z, \end{aligned}$$

where

$$(101)\qquad s_{ik} = \tfrac{1}{2}(3v_{ik} - t_{ik}).$$

Equations (100) show that the nessary and sufficient condition that S' be a pseudo-developable is

$$(102)\qquad F = s_{11}s_{22} - s_{12}s_{21} = 9\mathcal{D} + K - 3\mathcal{N} = 0.$$

For, S' is a pseudo-developable if and only if the tangent 4-space at P_ρ coincides with the tangent 4-space at P_σ. If these two 4-spaces coincide, the point given by the first equation of (100) in which the tangent 4-space at P_ρ meets L_{yz} must coincide with the point given by the second equation of (100) in which the tangent 4-space at P_σ meets the line L_{yz}. These points coincide if F = 0. Moreover, if F = 0, y and z can be eliminated from (100) so as to give an equation of the form

$$\lambda_1\rho'' + \lambda_2\rho' + \lambda_3\rho + \lambda_4\sigma'' + \lambda_5\sigma' + \lambda_6\sigma = 0.$$

This is the condition that S' be a pseudo-developable. The differential equation which the independent variable making F = 0 must satisfy is of the second order in η. <u>There are, therefore, ∞^3 surfaces S' which are pseudo-developables</u>.

We shall now proceed to find the differential equations for S'

under the assumption that F is not equal to zero. Let us denote by g_{ik} the quantities formed from the s_{ik}'s in the same way that the v_{ik}'s are formed from the u_{ik}'s, so that we have .

(103)
$$g_{ik} = s'_{ik} + \sum_{j=1}^{2} (s_{jk} p_{ij} - s_{ij} p_{jk}).$$

If we differentiate R and S and eliminate y', y, z', z from the resulting equations, we find

(104)
$$F\mathcal{R}' - Fs_{11}\rho - Fs_{12}\sigma + \tau_{11}\mathcal{R} + \tau_{12}S = 0,$$
$$FS' - Fs_{21}\rho - Fs_{22}\sigma + \tau_{21}\mathcal{R} + \tau_{22}S = 0,$$

where

(105)
$$\tau_{11} = Fp_{11} + s_{21}g_{12} - s_{22}g_{11},$$
$$\tau_{12} = Fp_{12} - s_{11}g_{12} + s_{12}g_{11},$$
$$\tau_{21} = Fp_{21} + s_{21}g_{22} - s_{22}g_{21},$$
$$\tau_{22} = Fp_{22} - s_{11}g_{22} + s_{12}g_{21}.$$

Let us put

(106)
$$s_{21}g_{12} - s_{22}g_{11} = 3F\lambda_{11},$$
$$-s_{11}g_{12} + s_{12}g_{11} = 3F\lambda_{12},$$
$$s_{21}g_{22} - s_{22}g_{21} = 3F\lambda_{21},$$
$$-s_{11}g_{22} + s_{12}g_{21} = 3F\lambda_{22},$$

whence

(107)
$$3(s_{11}\lambda_{11} + s_{21}\lambda_{12}) = -g_{11},$$
$$3(s_{12}\lambda_{11} + s_{22}\lambda_{12}) = -g_{12},$$
$$3(s_{11}\lambda_{21} + s_{21}\lambda_{22}) = -g_{21},$$
$$3(s_{12}\lambda_{21} + s_{22}\lambda_{22}) = -g_{22}.$$

Now if we introduce the values of R and S from(100), equations (104) become

$$(108)\qquad \begin{aligned} &\rho''' + 3\mathcal{P}_{11}\rho'' + 3\mathcal{P}_{12}\sigma'' + 3\mathcal{Q}_{11}\rho' + 3\mathcal{Q}_{12}\sigma' + \mathcal{R}_{11}\rho + \mathcal{R}_{12}\sigma = 0,\\ &\sigma''' + 3\mathcal{P}_{21}\rho'' + 3\mathcal{P}_{22}\sigma'' + 3\mathcal{Q}_{21}\rho' + 3\mathcal{Q}_{22}\sigma' + \mathcal{R}_{21}\rho + \mathcal{R}_{22}\sigma = 0, \end{aligned}$$

where

$$(109)\qquad \begin{aligned} &\mathcal{P}_{ik} = p_{ik} + \lambda_{ik},\\ &\mathcal{Q}_{ik} = q_{ik} + 2\sum_{j=1}^{2} p_{jk}\lambda_{ij},\\ &\mathcal{R}_{ik} = r_{ik} - 3v_{ik} + 3\sum_{j=1}^{2}\lambda_{ij}(q_{jk} - 2u_{jk}). \end{aligned}$$

Equations (108) constitute the system of form (A) which defines S'.

We can now determine S", the derivative of the first kind of S', the second derivative of the first kind of S. To find two points on the generator of S" corresponding to the generator $L_{\rho\sigma}$ of S', it is only necessary to form two quantities λ and τ from equations (108) in the same way that ρ and σ are formed from equations (A). The expressions for λ and τ are

$$(110)\qquad \begin{aligned} &\lambda = \rho' + \mathcal{P}_{11}\rho + \mathcal{P}_{12}\sigma = \rho' + (p_{11} + \lambda_{11})\rho + (p_{12} + \lambda_{12})\sigma,\\ &\tau = \sigma' + \mathcal{P}_{21}\rho + \mathcal{P}_{22}\sigma = \sigma' + (p_{21} + \lambda_{21})\rho + (p_{22} + \lambda_{22})\sigma. \end{aligned}$$

We have seen that two points on the generator of S_1' corresponding to $L_{\rho\sigma}$ are given by the quantities μ and ν where

$$\begin{aligned} &\mu = \rho' + p_{11}\rho + p_{12}\sigma,\\ &\nu = \sigma' + p_{21}\rho + p_{22}\sigma. \end{aligned}$$

Clearly two corresponding generators of S" and S_1' intersect if and only if

$$(111)\qquad \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 0.$$

Since F cannot be zero if we assume that S' is not a pseudo-developable,

we find on substituting the values of the λ_{ik}'s from (106) that the condition (111) is equivalent to the condition

(112)
$$g_{11} g_{22} - g_{12} g_{21} = 0.$$

The most general independent variable for which condition (112) is fulfilled is obtained by integrating a differential equation of the third order in η. Therefore, there exists ∞^3 surfaces S' whose derivatives of the first kind have the property that their generators intersect the corresponding generators of the derivative of the second kind of S.

Equations (110) also show that S" and S_1' coincide if and only if $\lambda_{ik} = 0$. The conditions $\lambda_{ik} = 0$ are equivalent to $g_{ik} = 0$. If these conditions are satisfied for any pair of curves on S, they are satisfied for every pair as long as the independent variable is not transformed.

Furthermore, without changing the independent variable we can make $p_{ik} = 0$. Then also will $P_{ik} = 0$, so that the pseudo-asymptotic curves of S and S' will correspond to each other. The conditions $g_{ik} = 0$ are now equivalent to the conditions r_{ik} = const. In other words, if the system (A) can be written in the form

(113)
$$\begin{aligned} y''' + 3 q_{11} y' + 3 q_{12} z' + r_{11} y + r_{12} z = 0, \\ z''' + 3 q_{21} y' + 3 q_{22} z' + r_{21} y + r_{22} z = 0, \end{aligned}$$

where the coefficients r_{ik} are constants, then the differential equations of S' are also in the semi-canonical form. The system for S' under these conditions is

(114)
$$\begin{aligned} \rho''' + 3 q_{11} \rho' + 3 q_{12} \sigma' + (r_{11} + 3q_{11}')\rho + (r_{12} + 3 q_{12}')\sigma = 0, \\ \sigma''' + 3 q_{21} \rho' + 3 q_{22} \sigma' + (r_{21} + 3q_{21}')\rho + (r_{22} + 3 q_{22}')\sigma = 0. \end{aligned}$$

If the q_{ik}'s are also constants, systems (113) and (114) are identical,

and S and S' are projective transformations of each other.

10. Significance of the covariants K_3 and C_6.

Let us consider the covariant

(115) $$K_3 = E_1' + (p_{11} + p_{22}) E_1,$$

where

$$E_1 = u_{12} z^2 - u_{21} y^2 + (u_{11} - u_{22}) y z.$$

If we introduce this value of E_1 into K_3 and substitute for y' and z' their values obtained from equations (57), we have

$$K_3 = z\{2u_{12}\sigma + (u_{11}-u_{22})\rho + \tfrac{1}{2}(v_{11}-v_{22}) + v_{12} z\}$$
$$- y\{2u_{21}\rho - (u_{11}-u_{22})\sigma - \tfrac{1}{2}(v_{11}-v_{22}) + v_{21} y\}.$$

It is clear that K_3 determines a ruled surface Σ whose generator corresponding to L_{yz} is the line joining the points P_α and P_β where

(116) $$\alpha = 2u_{12}\sigma + (u_{11}-u_{22})\rho + \tfrac{1}{2}(v_{11}-v_{22}) y + v_{12} z,$$
$$\beta = 2u_{21}\rho - (u_{11}-u_{22})\sigma - \tfrac{1}{2}(v_{11}-v_{22}) z + v_{21} y.$$

Since a transformation $\xi = \xi(x)$ of the independent variable converts α and β into $\bar{\alpha}$ and $\bar{\beta}$ where

$$\bar{\alpha} = \frac{1}{(\xi')^3}\alpha, \quad \bar{\beta} = \frac{1}{(\xi')^3}\beta,$$

the ~~same~~ surface Σ is left invariant, i.e., it is uniquely determined by S and does not in any way depend on the independent variable. As the points α and β are found to be transformed cogrediently with y and z by a transformation of the dependent variables, there is a one-to-one correspondence between the points of a generator of S and the corresponding generator of Σ.

It is easy to find a construction for the surface Σ under the assumption that $\Theta_4 \neq 0$. We can take the integral curves on S to be the two branches of C_u so that we have $u_{12} = u_{21} = 0$. By a proper choice of the independent variable we can make $u_{11} - u_{22} = 1$, and then we have $\Theta_4 = 1$. This particular selection of the independent variable determines a unique derivative surface of the first kind which we shall call <u>the</u> <u>princial derivative surface</u> <u>of</u> <u>S</u>. After these reductions have been made, the expressions for α and β become

$$
(117) \qquad \begin{aligned} \alpha &= \rho - p_{12} z = y' + p_{11} y, \\ \beta &= -\sigma + p_{21} y = -z' - p_{22} z. \end{aligned}
$$

It is evident from these equations that P_α is the point of intersection of the line $P_\rho P_z$ with the tangent line to C_y at P_y, while P_β is the point of intersection of the line $P_\sigma P_y$ with the tangent line to C_z at P_z. The generator of the surface Σ is thus completely determined.

The covariant

$$
(118) \qquad C_6 = 4 \mathcal{P}_2 \Theta_4 + \mathcal{P}_1 \Theta_4' = 4(z\mu - y\nu)\Theta_4 + \Theta_4'(z\rho - y\sigma)
$$

also determines a system of ruled surfaces associated with S. Of course it is necessary to assume that Θ_4 does not vanish.

Now C_6 can be written in the form

$$
C_6 = z(4\Theta_4 \mu + \Theta_4' \rho) - y(4\Theta_4 \nu + \Theta_4' \sigma).
$$

Let us consider the ruled surface T whose generator is the line joining P_γ and P_δ where

$$
(119) \qquad \begin{aligned} \gamma &= 4\Theta_4 \mu + \Theta_4' \rho, \\ \delta &= 4\Theta_4 \nu + \Theta_4' \sigma. \end{aligned}
$$

The quantities γ and δ are cogredient with y and z respectively under

a transformation of the dependent variables, but a transformation $\xi = \xi(x)$ of the independent variable converts γ and δ into $\bar{\gamma}$ and $\bar{\delta}$ where

$$(120) \qquad \begin{aligned} \bar{\gamma} &= \frac{1}{(\xi')^6}\{\gamma + (4\eta'\Theta_4 + \eta\Theta_4' - 4\eta^2\Theta_4)y\}, \\ \bar{\delta} &= \frac{1}{(\xi')^6}\{\delta + (4\eta'\Theta_4 + \eta\Theta_4' - 4\eta^2\Theta_4)z\}. \end{aligned}$$

As $P_{\bar{\gamma}}$ is a point on the line joining P_y and P_γ, and $P_{\bar{\delta}}$ is a point on the line joining P_z and P_δ, it is clear that the generator $L_{\gamma\delta}$ merely moves along two fixed lines L_1 and L_2 through P_y and P_z respectively when the independent variable is transformed.

We have seen that, when the independent variable is so chosen that $\Theta_4 = 1$, S' becomes the principal derivative surface. In that case P_γ coincides with P_μ and P_δ with P_ν, so that T coincides with S_1'. Hence, L_1 is the line joining P_y and the corresponding point of the derivative of the second kind associated with the principal derivative surface. The line L_2 is determined similarly. Now, let us take two pseudo-asymptotic curves for the integral curves on S. The line L_1 and the points P_ρ and P_μ all lie in the osculating plane of C_y at P_y for any value of the independent variable. The intersection of L_1 and the line joining P_ρ and P_μ, which is the tangent line to C_ρ at P_ρ, is the point P_γ. Likewise the intersection of L_2 and the tangent to C_σ at P_σ is the point P_δ. The line joining P_γ and P_δ is the generator of T corresponding to L_{yz}.